

# Almost global solutions of semilinear wave equations with the critical exponent in high dimensions \*

In memory of Professor Rentaro Agemi

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**Keywords:** semilinear wave equation, high dimensions, critical exponent, lifespan

**MSC2010:** primary 35L71, 35E15, secondary 35A01, 35A09, 35B33, 35B44

## Abstract

We are interested in the “almost” global-in-time existence of classical solutions in the general theory for nonlinear wave equations. All the three such cases are known to be sharp due to blow-up results in the critical case for model equations. However, it is known that we have a possibility to get the global-in-time existence for two of them in low space dimensions if the nonlinear term is of derivatives of the unknown function and satisfies so-called null condition, or non-positive condition. But another one for the quadratic term in four space dimensions is out of the case as the nonlinear term should include a square of the unknown function itself.

In this paper, we get one more example guaranteeing the sharpness of the almost global-in-time existence in four space dimensions. It is also the first example of the blow-up of classical solutions for non-single and indefinitely signed term in high dimensions. Such a term arises from the neglect of derivative-loss factors in Duhamel’s formula for positive and single nonlinear term. This fact may help us to describe a criterion to get the global-in-time existence in this critical situation.

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\*This work is partially supported by the Grant-in-Aid for Scientific Research (C) (No. 24540183), Japan Society for the Promotion of Science.

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# 1 Introduction

First we shall outline the general theory on the initial value problem for fully nonlinear wave equations,

$$\begin{cases} u_{tt} - \Delta u = H(u, Du, D_x Du) & \text{in } \mathbf{R}^n \times [0, \infty), \\ u(x, 0) = \varepsilon f(x), \quad u_t(x, 0) = \varepsilon g(x), \end{cases} \quad (1.1)$$

where  $u = u(x, t)$  is a scalar unknown function of space-time variables,

$$\begin{aligned} Du &= (u_{x_0}, u_{x_1}, \dots, u_{x_n}), \quad x_0 = t, \\ D_x Du &= (u_{x_i x_j}, \quad i, j = 0, 1, \dots, n, \quad i + j \geq 1), \end{aligned}$$

$f, g \in C_0^\infty(\mathbf{R}^n)$  and  $\varepsilon > 0$  is a “small” parameter. We note that it is impossible to construct a general theory for “large”  $\varepsilon$  due to blow-up results. For example, see Glassey [8], Levine [21], or Sideris [31]. Let

$$\widehat{\lambda} = (\lambda; (\lambda_i), i = 0, 1, \dots, n; (\lambda_{ij}), i, j = 0, 1, \dots, n, \quad i + j \geq 1).$$

Suppose that the nonlinear term  $H = H(\widehat{\lambda})$  is a sufficiently smooth function with

$$H(\widehat{\lambda}) = O(|\widehat{\lambda}|^{1+\alpha})$$

in a neighborhood of  $\widehat{\lambda} = 0$ , where  $\alpha \geq 1$  is an integer. Let us define the lifespan  $\widetilde{T}(\varepsilon)$  of classical solutions of (1.1) by

$$\widetilde{T}(\varepsilon) = \sup\{t > 0 : \exists \text{ a classical solution } u(x, t) \text{ of (1.1) for arbitrarily fixed data, } (f, g).\}.$$

When  $\widetilde{T}(\varepsilon) = \infty$ , the problem (1.1) admits a global-in-time solution, while we only have a local-in-time solution on  $[0, \widetilde{T}(\varepsilon))$  when  $\widetilde{T}(\varepsilon) < \infty$ . For local-in-time solutions, one can measure the long time stability of a zero solution by orders of  $\varepsilon$ . Because the uniqueness of the solution of (1.1) may yield that  $\lim_{\varepsilon \rightarrow +0} \widetilde{T}(\varepsilon) = \infty$ . Such an uniqueness theorem can be found in Appendix of John [15] for example. From now on, we omit “-in-time” and simply use “global” and “local”.

In Chapter 2 of Li and Chen [23], we have long histories on the estimate for  $\widetilde{T}(\varepsilon)$ . The lower bounds of  $\widetilde{T}(\varepsilon)$  are summarized in the following table. Let  $a = a(\varepsilon)$  satisfy

$$a^2 \varepsilon^2 \log(a + 1) = 1 \quad (1.2)$$

and  $c$  stands for a positive constant independent of  $\varepsilon$ . Then, due to the fact that it is impossible to obtain an  $L^2$  estimate for  $u$  itself by standard energy methods, we have

$\tilde{T}(\varepsilon) \geq$	$\alpha = 1$	$\alpha = 2$	$\alpha \geq 3$
$n = 2$	$ca(\varepsilon)$ in general case, $c\varepsilon^{-1}$ if $\int_{\mathbf{R}^2} g(x)dx = 0$ , $c\varepsilon^{-2}$ if $\partial_u^2 H(0) = 0$	$c\varepsilon^{-6}$ in general case, $c\varepsilon^{-18}$ if $\partial_u^3 H(0) = 0$ , $\exp(c\varepsilon^{-2})$ if $\partial_u^3 H(0) = \partial_u^4 H(0) = 0$	$\infty$
$n = 3$	$c\varepsilon^{-2}$ in general case, $\exp(c\varepsilon^{-1})$ if $\partial_u^2 H(0) = 0$	$\infty$	$\infty$
$n = 4$	$\exp(c\varepsilon^{-2})$ in general case, $\infty$ if $\partial_u^2 H(0) = 0$	$\infty$	$\infty$
$n \geq 5$	$\infty$	$\infty$	$\infty$

The result for  $n = 1$  is that

$$\tilde{T}(\varepsilon) \geq \begin{cases} \varepsilon^{-\alpha/2} & \text{in general case,} \\ c\varepsilon^{-\alpha(1+\alpha)/(2+\alpha)} & \text{if } \int_{\mathbf{R}} g(x)dx = 0, \\ c\varepsilon^{-\alpha} & \text{if } \partial_u^\beta H(0) = 0 \text{ for } 1 + \alpha \leq \forall \beta \leq 2\alpha. \end{cases} \quad (1.3)$$

For references on these results, see Li and Chen [23]. We shall skip to refer them here. But we note that two parts in this table are different from the one in Li and Chen [23]. One is the general case in  $(n, \alpha) = (4, 1)$ . In this part, the lower bound of  $\tilde{T}(\varepsilon)$  is  $\exp(c\varepsilon^{-1})$  in Li and Chen [23]. But later, it has been improved by Li and Zhou [24]. Another is the case for  $\partial_u^3 H(0) = 0$  in  $(n, \alpha) = (2, 2)$ . This part is due to Katayama [17]. But it is missing in Li and Chen [23]. Its reason is closely related to the sharpness of results in the general theory. The sharpness is achieved by the fact that there is no possibility to improve the lower bound of  $\tilde{T}(\varepsilon)$  in sense of order of  $\varepsilon$  by blow-up results for special equations and special data. It is expressed in the upper bound of  $\tilde{T}(\varepsilon)$  with the same order of  $\varepsilon$  as in the lower bound. On this matter, Li and Chen [23] says that all these lower bounds are known to be sharp except for  $(n, \alpha) = (4, 1)$ . But before this article, Li [22] says that  $(n, \alpha) = (2, 2)$  has also open sharpness while the case for  $\partial_u^3 H(0) = 0$  is still missing. Li and Chen [23] might have dropped the open sharpness in  $(n, \alpha) = (2, 2)$  by conjecture that  $\partial_u^4 H(0) = 0$  is a technical condition. No one disagrees with this observation because the model case of  $H = u^4$

has a global solution in two space dimensions,  $n = 2$ . See the next section below. However, Zhou and Han [40] have obtained this final sharpness in  $(n, \alpha) = (2, 2)$  by studying  $H = u_t^2 u + u^4$ . This puts Katayama's result into the table after 20 years from Li and Chen [23]. We note that Godin [11] has showed the sharpness of the case for  $\partial_u^3 H(0) = \partial_u^4 H(0) = 0$  in  $(n, \alpha) = (2, 2)$  by studying  $H = u_t^3$ . This result has reproved by Zhou and Han [39].

We now turn back to another open sharpness of the general case in  $(n, \alpha) = (4, 1)$ . It has been obtained by our previous work, Takamura and Wakasa [33], by studying model case of  $H = u^2$ . This part had been open more than 20 years in the analysis on the critical case for model equations,  $u_{tt} - \Delta u = |u|^p$  ( $p > 1$ ). We mention to whole histories on this equation precisely in the next section. In this way, the general theory and its optimality have been completed.

After the completion of the general theory, we are interested in “almost” global existence, namely, the case where  $\tilde{T}(\varepsilon)$  has an lower bound of the exponential function of  $\varepsilon$  with a negative power. Such a case appears in  $(n, \alpha) = (2, 2), (3, 1), (4, 1)$  in the table of the general theory. It is remarkable that Klainerman [19] and Christodoulou [4] have independently found a special structure on  $H = H(Du, D_x Du)$  in  $(n, \alpha) = (3, 1)$  which guarantees the global existence. This algebraic condition on nonlinear terms of derivatives of the unknown function is so-called “null condition”. It has been also established independently by Godin [11] for  $H = H(Du)$  and Katayama [16] for  $H = H(Du, D_x Du)$  in  $(n, \alpha) = (2, 2)$ . The null condition had been supposed to be not sufficient for the global existence in  $(n, \alpha) = (2, 2)$ . For this direction, Agemi [1] proposed “non-positive condition” in this case for  $H = H(Du)$ . This conjecture has been verified by Hoshiga [12] and Kubo [20] independently. It might be necessary and sufficient condition to the global existence. On the other hand, the situation in  $(n, \alpha) = (4, 1)$  is completely different from  $(n, \alpha) = (2, 2), (3, 1)$  because  $H$  has to include  $u^2$ .

In this paper, we get the first attempt to clarify a criterion on  $H$  guaranteeing the global existence by showing another blow-up example of  $H$ . More precisely, we have an almost global existence and its optimality for an equation of the form

$$\begin{aligned} u_{tt} - \Delta u = u^2 & - \frac{1}{\pi^2} \int_0^t d\tau \int_{|\xi| \leq 1} \frac{(u_t u)(x + (t - \tau)\xi, \tau)}{\sqrt{1 - |\xi|^2}} d\xi \\ & - \frac{\varepsilon^2}{2\pi^2} \int_{|\xi| \leq 1} \frac{f(x + t\xi)^2}{\sqrt{1 - |\xi|^2}} d\xi \end{aligned} \quad (1.4)$$

in  $\mathbf{R}^4 \times [0, \infty)$ . We note that the third term in the right-hand side of (1.4) can be neglected by simple modification. One can say that this result is the first

example of the blowing-up of a classical solution to nonlinear wave equation with “non-single” and “indefinitely signed” term in high dimensions. (1.4) arises from a neglect of derivative loss factors in Duhamel’s term for positive and single nonlinear term,  $u^2$ . We introduce this observation in the next section with more general situation about on space dimensions. Therefore, one can conclude that derivative loss factors in Duhamel’s term due to high dimensions do not contribute to any order of  $\varepsilon$  in the estimate of the lifespan.

Finally, we note that, in contrast with (1.4), another equation of the form

$$\begin{aligned} u_{tt} - \Delta u = u^2 & - \frac{1}{2\pi^2} \int_0^t d\tau \int_{|\omega|=1} (u_t u)(x + (t - \tau)\omega, \tau) dS_\omega \\ & - \frac{\varepsilon}{4\pi^2} \int_{|\omega|=1} (\varepsilon f^2 + \Delta f + 2\omega \cdot \nabla g)(x + t\omega) dS_\omega \end{aligned}$$

admits a global classical solution in  $\mathbf{R}^4 \times [0, \infty)$ . Its details will appear in our forthcoming paper.

This paper is organized as follows. Main theorems and whole histories on closely related model equations,  $u_{tt} - \Delta u = |u|^p$ , are stated in the next section including how to derive our problem. They are discussed in all high space dimensions and for the nonlinear term with fractional powers to describe what the critical power depends on. In section 3, we introduce a weighted  $L^\infty$  space in which the solution will be constructed by contraction mapping. In section 4, we show *a priori* estimate for the existence part. The lower bounds of the lifespan in odd space dimensions or even space dimensions are obtained in section 5 or section 6 respectively. Upper bounds of the lifespan in odd space dimensions are obtained in section 7 for the critical case and in section 8 for the subcritical case. Similarly to them, upper bounds of the lifespan in even space dimensions are obtained in section 9 for the critical case and in section 10 for the subcritical case.

The essential part of this work has been completed when the second author was in the 2nd year of the master course, Graduate School of Systems Information Science, Future University Hakodate.

## 2 Model equations and main theorems

Before deriving our problem (1.4), we shall introduce the whole histories on the following model problems. By such problems, the optimality of the general theory is guaranteed especially in the case where nonlinear term  $H$  includes the lower order of  $u$  itself. This may help us to know the difficulty to analyze the quadratic terms in four space dimensions.

We first consider an initial value problem,

$$\begin{cases} u_{tt} - \Delta u = |u|^p, & \text{in } \mathbf{R}^n \times [0, \infty), \\ u(x, 0) = \varepsilon f(x), \quad u_t(x, 0) = \varepsilon g(x) \end{cases} \quad (2.1)$$

assuming that  $\varepsilon > 0$  is “small” again. Let us define a lifespan  $T(\varepsilon)$  of a solution of (2.1) by

$$T(\varepsilon) = \sup\{t > 0 : \exists \text{ a solution } u(x, t) \text{ of (2.1) for arbitrarily fixed } (f, g)\},$$

where “solution” means classical one when  $p \geq 2$ . When  $1 < p < 2$ , it means weak one, but sometimes the one of a solution of associated integral equations to (2.1) by standard Strichartz’s estimate. See Georgiev, Takamura and Zhou [7] for example on such an argument.

When  $n = 1$ , we have  $T(\varepsilon) < \infty$  for any power  $p > 1$  by Kato [18]. When  $n \geq 2$ , we have the following Strauss’ conjecture on (2.1) by Strauss [32].

$$\begin{aligned} T(\varepsilon) &= \infty && \text{if } p > p_0(n) \text{ and } \varepsilon \text{ is “small”} && \text{(global-in-time existence),} \\ T(\varepsilon) &< \infty && \text{if } 1 < p \leq p_0(n) && \text{(blow-up in finite time),} \end{aligned}$$

where  $p_0(n)$  is so-called Strauss’ exponent defined by positive root of the quadratic equation,

$$\gamma(p, n) = 2 + (n + 1)p - (n - 1)p^2 = 0. \quad (2.2)$$

That is,

$$p_0(n) = \frac{n + 1 + \sqrt{n^2 + 10n - 7}}{2(n - 1)}. \quad (2.3)$$

We note that  $p_0(n)$  is monotonously decreasing in  $n$  and  $p_0(4) = 2$ . This conjecture had been verified by many authors of partial results. All the references on the final result in each part can be summarized in the following table.

	$p < p_0(n)$	$p = p_0(n)$	$p > p_0(n)$
$n = 2$	Glassey [9]	Schaeffer [29]	Glassey [10]
$n = 3$	John [14]	Schaeffer [29]	John [14]
$n \geq 4$	Sideris [30]	Yordanov & Zhang [34] Zhou [38], indep.	Georgiev & Lindblad & Sogge [6]

In the blow-up case of  $1 < p \leq p_0(n)$ , we are interested in the estimate of the lifespan  $T(\varepsilon)$ . From now on,  $c$  and  $C$  stand for positive constants but

independent of  $\varepsilon$ . When  $n = 1$ , we have the following estimate of the lifespan  $T(\varepsilon)$  for any  $p > 1$ .

$$\begin{cases} c\varepsilon^{-(p-1)/2} \leq T(\varepsilon) \leq C\varepsilon^{-(p-1)/2} & \text{if } \int_{\mathbf{R}} g(x)dx \neq 0, \\ c\varepsilon^{-p(p-1)/(p+1)} \leq T(\varepsilon) \leq C\varepsilon^{-(p-1)/(p+1)} & \text{if } \int_{\mathbf{R}} g(x)dx = 0. \end{cases} \quad (2.4)$$

This result has been obtained by Y.Zhou [35]. We note that its order of  $\varepsilon$  coincides with the general theory when  $p = 1 + \alpha$  ( $\alpha = 1, 2, 3, \dots$ ). Moreover, Lindblad [25] has obtained more precise result for  $p = 2$ ,

$$\begin{cases} \exists \lim_{\varepsilon \rightarrow +0} \varepsilon^{1/2} T(\varepsilon) > 0 & \text{for } \int_{\mathbf{R}} g(x)dx \neq 0, \\ \exists \lim_{\varepsilon \rightarrow +0} \varepsilon^{2/3} T(\varepsilon) > 0 & \text{for } \int_{\mathbf{R}} g(x)dx = 0. \end{cases} \quad (2.5)$$

Similarly to this, Lindblad [25] has also obtained the following result for  $(n, p) = (2, 2)$ .

$$\begin{cases} \exists \lim_{\varepsilon \rightarrow +0} a(\varepsilon)^{-1} T(\varepsilon) > 0 & \text{for } \int_{\mathbf{R}^2} g(x)dx \neq 0 \\ \exists \lim_{\varepsilon \rightarrow +0} \varepsilon T(\varepsilon) > 0 & \text{for } \int_{\mathbf{R}^2} g(x)dx = 0, \end{cases} \quad (2.6)$$

where  $a(\varepsilon)$  is the one in (1.2).

When  $1 < p < p_0(n)$  ( $n \geq 3$ ) or  $2 < p < p_0(2)$  ( $n = 2$ ), we have the following conjecture.

$$c\varepsilon^{-2p(p-1)/\gamma(p,n)} \leq T(\varepsilon) \leq C\varepsilon^{-2p(p-1)/\gamma(p,n)}, \quad (2.7)$$

where  $\gamma(p, n)$  is defined by (2.2). We note that (2.7) coincides with the second line in (2.4) if we define  $\gamma(p, n)$  by (2.2) even for  $n = 1$ . All the results verifying this conjecture are summarized in the following table.

	lower bound of $T(\varepsilon)$	upper bound of $T(\varepsilon)$
$n = 2$	Zhou [37]	Zhou [37]
$n = 3$	Lindblad [25]	Lindblad [25]
$n \geq 4$	Lindblad & Sogge [26] : $n \leq 8$ or radially symmetric sol.	Sideris [30]

We note that, for  $n = 2, 3$ ,

$$\exists \lim_{\varepsilon \rightarrow +0} \varepsilon^{2p(p-1)/\gamma(p,n)} T(\varepsilon) > 0.$$

is established. Moreover, the upper bound in the case where  $n \geq 4$  easily follows from the rescaling method applied to the proof in Sideris [30] which proves  $T(\varepsilon) < \infty$ . Such an argument can be found in Georgiev, Takamura and Zhou [7].

On the other hand, when  $p = p_0(n)$ , we have the following conjecture.

$$\exp(c\varepsilon^{-p(p-1)}) \leq T(\varepsilon) \leq \exp(C\varepsilon^{-p(p-1)}). \quad (2.8)$$

All the results verifying this conjecture are also summarized in the following table.

	lower bound of $T(\varepsilon)$	upper bound of $T(\varepsilon)$
$n = 2$	Zhou [37]	Zhou [37]
$n = 3$	Zhou [36]	Zhou [36]
$n \geq 4$	Lindblad & Sogge [26] : $n \leq 8$ or radially symm. sol.	Takamura & Wakasa [33]

In this way, we realize that one of the last open problem had been the upper bound of  $T(\varepsilon)$  for the critical case  $p = p_0(n)$  in high dimensions  $n \geq 4$ . This difficulty is due to the lack of the positivity of the fundamental solution of linear wave equations which is caused by so-called “derivative loss”.

We are now in a position to derive our problem (1.4) in the general situation by neglecting such derivative loss factors. From now on, we assume that  $n \geq 2$  and write  $n = 2m$ ,  $2m + 1$  ( $m = 1, 2, 3, \dots$ ). Let us consider the following initial value problem.

$$\begin{cases} u_{tt} - \Delta u = F(u) & \text{in } \mathbf{R}^n \times [0, \infty), \\ u(x, 0) = \varepsilon f(x), \quad u_t(x, 0) = \varepsilon g(x), \end{cases} \quad (2.9)$$

where  $f \in C^{m+3}(\mathbf{R}^n)$ ,  $g \in C^{m+2}(\mathbf{R}^n)$  and  $F \in C^{m+1}(\mathbf{R})$ . Then, any solution  $u$  of (2.9) has to satisfy

$$u(x, t) = \varepsilon u^0(x, t) + \int_0^t R(F(u(\cdot, \tau))|x, t - \tau) d\tau. \quad (2.10)$$

Here we set

$$u^0(x, t) = \partial_t R(f|x, t) + R(g|x, t)$$

and

$$R(\phi|x, t) = \frac{1}{(2m-1)!!} \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{m-1} \{ t^{2m-1} M(\phi|x, t) \},$$

where

$$M(\phi|x, r) = \begin{cases} \frac{1}{\omega_n} \int_{|\omega|=1} \phi(x + r\omega) dS_\omega & \text{for } n = 2m + 1, \\ \frac{2}{\omega_{n+1}} \int_{|\xi| \leq 1} \frac{\phi(x + r\xi)}{\sqrt{1 - |\xi|^2}} d\xi & \text{for } n = 2m. \end{cases} \quad (2.11)$$



We note that  $\omega_n$  stands for a measure of the unit sphere in  $\mathbf{R}^n$ , i.e.

$$\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)} = \begin{cases} \frac{2(2\pi)^m}{(2m-1)!!} & \text{for } n = 2m+1, \\ \frac{2\pi^m}{(m-1)!} & \text{for } n = 2m. \end{cases}$$

This representation formula is well-known. See pp.681-692 in Courant and Hilbert [5] for example.

If we neglect derivative loss factors, namely all the derivatives of  $F(u)$ , from the Duhamel's term in (2.10), then we get a new integral equation,

$$u(x, t) = \varepsilon u^0(x, t) + L(F(u))(x, t) \quad \text{for } (x, t) \in \mathbf{R}^n \times [0, \infty), \quad (2.12)$$

where we set

$$L(F(u))(x, t) = \frac{1}{2m-1} \int_0^t (t-\tau) M(F(u(\cdot, \tau))|x, t-\tau) d\tau. \quad (2.13)$$

A simple computation yields that if  $u$  is a  $C^2$  solution of (2.12) with  $f \in C^{m+3}(\mathbf{R}^n)$ ,  $g \in C^{m+2}(\mathbf{R}^n)$  and  $F \in C^2(\mathbf{R})$ , then  $u$  satisfies

$$\begin{cases} u_{tt} - \Delta u = F(u) - G(x, t) \\ \quad - \frac{2(m-1)}{2m-1} M(F(\varepsilon f)|x, t) & \text{in } \mathbf{R}^n \times [0, \infty), \\ u(x, 0) = \varepsilon f(x), \quad u_t(x, 0) = \varepsilon g(x), & x \in \mathbf{R}^n \end{cases} \quad (2.14)$$

in the classical sense, where we set

$$G(x, t) = \frac{2(m-1)}{2m-1} \int_0^t M(F'(u(\cdot, \tau))u_t(\cdot, \tau)|x, t-\tau) d\tau. \quad (2.15)$$

In this way, (1.4) follows from setting  $n = 4$  ( $m = 2$ ) and  $F(u) = u^2$  in (2.14).

**Remark 2.1** *The uniqueness of the solution of (2.14) is open while Agemi, Kubota and Takamura [2] has a wrong comment on this fact after (1.8) on 242p. in [2]. The restricted uniqueness theorem such as in Appendix 1 in John [15] cannot be applicable here because (99a) in [15] does not hold for this case.*

Moreover, assuming  $F \in C^{m+1}(\mathbf{R})$  if  $f \not\equiv 0$ , and replacing  $\varepsilon u^0(x, t)$  by a classical solution  $v = v(x, t)$  of

$$\begin{cases} v_{tt} - \Delta v = \frac{2(m-1)}{2m-1} M(F(\varepsilon f)|x, t) & \text{in } \mathbf{R}^n \times [0, \infty), \\ v(x, 0) = \varepsilon f(x), \quad v_t(x, 0) = \varepsilon g(x) & x \in \mathbf{R}^n \end{cases} \quad (2.16)$$

in (2.12), we have that a  $C^2$  solution  $u$  of

$$u(x, t) = v(x, t) + L(F(u))(x, t) \quad \text{for } (x, t) \in \mathbf{R}^n \times [0, \infty) \quad (2.17)$$

satisfies

$$\begin{cases} u_{tt} - \Delta u = F(u) - G(x, t) & \text{in } \mathbf{R}^n \times [0, \infty), \\ u(x, 0) = \varepsilon f(x), \quad u_t(x, 0) = \varepsilon g(x), & x \in \mathbf{R}^n \end{cases} \quad (2.18)$$

in the classical sense. We note that  $G \equiv 0$  when  $n = 2, 3$  ( $m = 1$ ). This observation appears in pp.254-255 of Agemi, Kubota and Takamura [2].

From now on, in order to clarify the critical situation in (2.14) and (2.18), we may set  $F(u) = |u|^p$ , or  $|u|^{p-1}u$  ( $p > 1$ ). But, as we see, the existence of a classical solution  $v$  of (2.16) with  $f \not\equiv 0$  is guaranteed by additional regularity  $F \in C^{m+1}$ . In such case we have to regularize  $F(u)$  at  $u = 0$ . Let us define a lifespan  $\overline{T}(\varepsilon)$  by

$$\overline{T}(\varepsilon) = \sup\{t > 0 : \exists \text{ a solution } u(x, t) \text{ of (2.12) or (2.17) for arbitrarily fixed data, } (f, g).\},$$

where “solution” means a  $C^2$  solution for  $p \geq 2$ , or the  $C^1$  solution for  $1 < p < 2$ . Agemi, Kubota and Takamura [2] have obtained that  $\overline{T}(\varepsilon) = \infty$  for  $p > p_0(n)$  if  $\varepsilon$  is small enough, where  $p_0(n)$  is Strauss’ exponent. Their theorem is written for (2.17) only, but it is trivial to be available also for (2.12).

Our purpose in this paper is to establish the same results for  $\overline{T}(\varepsilon)$  as in (2.7) and (2.8) when  $n \geq 4$  and  $1 < p \leq p_0(n)$ . They are divided into two theorems below. We note that one can expect to get a  $C^2$  solution only for  $n = 4$  and  $p = p_0(4) = 2$  in this situation. Except for this case, we assume on  $F$  that

$$\begin{cases} \text{there exists a constant } A > 0 \text{ such that} \\ F \in C^1(\mathbf{R}) \text{ satisfies that } F(0) = F'(0) = 0 \text{ and} \\ |F'(s) - F'(\tilde{s})| \leq pA|s - \tilde{s}|^{p-1} \text{ for } 1 < p < 2. \end{cases} \quad (2.19)$$

Note that (2.19) implies that  $|F(s)| \leq A|s|^p$  for  $1 < p < 2$ . We also assume on the data that

$$\begin{cases} \text{both } f \in C_0^{m+3}(\mathbf{R}^n) \text{ and } g \in C_0^{m+2}(\mathbf{R}^n) \text{ do not} \\ \text{vanish identically and have compact support} \\ \text{contained in } \{x \in \mathbf{R}^n : |x| \leq k\} \text{ with some constant } k > 0. \end{cases} \quad (2.20)$$

Then, we have the following existence theorem for large time interval.

**Theorem 2.1** *Let  $n \geq 4$  and  $1 < p \leq p_0(n)$ . Assume  $F(s) = As^2$  when  $n = 4$  and  $p = p_0(4) = 2$  or (2.19) otherwise, where  $A$  is a positive constant. Suppose that (2.20) is fulfilled. Moreover, assume that  $F \in C^{m+1}(\mathbf{R})$  for (2.17). Then there exists a positive constant  $\varepsilon_0 = \varepsilon_0(f, g, n, p, k)$  such that the lifespan  $\bar{T}(\varepsilon)$  satisfies*

$$\begin{aligned} \bar{T}(\varepsilon) &\geq c\varepsilon^{-2p(p-1)/\gamma(p,n)} && \text{if } 1 < p < p_0(n), \\ \bar{T}(\varepsilon) &\geq \exp(c\varepsilon^{-p(p-1)}) && \text{if } p = p_0(n) \end{aligned} \quad (2.21)$$

for any  $\varepsilon$  with  $0 < \varepsilon \leq \varepsilon_0$ , where  $c$  is a positive constant independent of  $\varepsilon$ .

This theorem follows from the similar weighted  $L^\infty$  iteration method to Agemi, Kubota and Takamura [2]. Its basic argument has been introduced by John [14] in the simplest case for  $n = 3$ . The proof is divided into sections 3, 4, 5, 6 and 7 below.

For the counter part, the following assumptions on the data are required.

$$\left\{ \begin{array}{l} \text{Let } f \equiv 0, g(x) = g(|x|) \text{ and } g \in C_0^1([0, \infty)) \text{ satisfy that there} \\ \text{exist positive constants } k_0 \text{ and } k_1 \text{ with } 0 < k_0 < k_1 < k \\ \text{such that the following three conditions hold.} \\ \text{(i) } \text{supp } g \subset \{x \in \mathbf{R}^n : |x| \leq k\} \\ \text{(ii) } g(|x|) \geq 0 \text{ for } k_0 < |x| < k \text{ and } \int_{(k_1+k)/2}^k \lambda^{[n/2]} g(\lambda) d\lambda > 0, \\ \text{(iii) } k_0 \text{ is sufficiently close to } k \text{ to satisfy} \\ \quad P_m(z) > \frac{1}{2} \text{ and } T_m(z) > \frac{1}{2} \text{ for all } z > \frac{k_0}{k}, \\ \quad \text{where } P_m \text{ or } T_m \text{ denote Legendre or Tschbyscheff} \\ \quad \text{polynomials of degree } m \text{ respectively.} \end{array} \right. \quad (2.22)$$

Then, we have the following blow-up theorem.

**Theorem 2.2** *Let  $n \geq 4$  and  $F(u) = |u|^p$  with  $1 < p \leq p_0(n)$ . Assume that (2.22). Then there exist a positive constant  $\varepsilon_1 = \varepsilon_1(g, n, p, k)$  such that the lifespan  $\bar{T}(\varepsilon)$  satisfies*

$$\begin{aligned} \bar{T}(\varepsilon) &\leq C\varepsilon^{-2p(p-1)/\gamma(p,n)} && \text{if } 1 < p < p_0(n), \\ \bar{T}(\varepsilon) &\leq \exp(C\varepsilon^{-p(p-1)}) && \text{if } p = p_0(n) \end{aligned} \quad (2.23)$$

for any  $\varepsilon$  with  $0 < \varepsilon \leq \varepsilon_1$ , where  $C$  is a positive constant independent of  $\varepsilon$ .

The proof of this theorem is an iteration argument of point-wise estimates which is basically introduced by John [14] for  $n = 3$ . But for the critical case, we have to reduce the proof to the argument of Zhou [36] which compares

the solution with a blowing-up solution of nonlinear ordinary differential equation of the second order. We also have to employ the slicing method of the blow-up set which is introduced by Agemi, Kurokawa and Takamura [3] due to technical difficulties in high dimensions. See sections 7, 8, 9 and 10 below.

### 3 Weighted $L^\infty$ space

First, we shall state the following two lemmas on  $v$  which play key roles in proofs of Theorem 2.1 and Theorem 2.2. The first one is Huygens' principle in odd space dimensions.

**Lemma 3.1 (Agemi, Kubota and Takamura [2])** *Let  $n = 5, 7, 9, \dots$ . Under the same assumption as in Theorem 2.1, there exists a classical solution of (2.16) which satisfies*

$$\text{supp } v \subset \{x \in \mathbf{R}^n : t - k \leq |x| \leq t + k\}. \quad (3.1)$$

See 253p. in [2] for the proof of this lemma.

Next, we shall introduce the decay estimate for  $v$ . First, we write  $v$  in the form

$$v = v_0 + v_1. \quad (3.2)$$

Here,  $v_0 = \varepsilon u^0$  which is a linear part of (2.10) and  $v_1$  is a solution to the inhomogeneous wave equation

$$\begin{cases} (v_1)_{tt} - \Delta v_1 = \frac{2(m-1)}{2m-1} M(F(\varepsilon f)|x, t) & \text{in } \mathbf{R}^n \times [0, \infty), \\ v_1(x, 0) = (v_1)_t(x, 0) = 0, & x \in \mathbf{R}^n \end{cases} \quad (3.3)$$

where  $M$  is defined in (2.11). Then we have the following lemma.

**Lemma 3.2 (Agemi, Kubota and Takamura [2])** *Under the same assumption as in Theorem 2.1, there exists a positive constant  $C_{n,k,f,g}$  depending only on  $n, k, f$  and  $g$  such that  $v_0$  and  $v_1$  satisfies*

$$\sum_{|\alpha| \leq 1} |\nabla_x^\alpha v_0(x, t)| \leq \frac{C_{n,k,f,g} \varepsilon}{(t + |x| + 2k)^{(n-1)/2}} \quad (3.4)$$

and

$$\sum_{|\alpha| \leq 1} |\nabla_x^\alpha v_1(x, t)| \leq \frac{C_{n,k,f,p} \varepsilon^p}{(t + |x| + 2k)^{(n-1)/2}} \quad (3.5)$$

when  $n = 5, 7, 9, \dots$ , or

$$\sum_{|\alpha| \leq 2} |\nabla_x^\alpha v_0(x, t)| \leq \frac{C_{n,k,f,g\varepsilon}}{(t + |x| + 2k)^{(n-1)/2} (t - |x| + 2k)^{(n-1)/2}} \quad (3.6)$$

and

$$\sum_{|\alpha| \leq 2} |\nabla_x^\alpha v_1(x, t)| \leq \frac{C_{n,k,f^p\varepsilon^p}}{(t + |x| + 2k)^{(n-1)/2} (t - |x| + 2k)^{(n-3)/2}} \quad (3.7)$$

when  $n = 4, 6, 8, \dots$ , where  $C_{n,k,f^p}$  is a non-negative constant depending on  $n, k, f^p$  with  $C_{n,k,f^p} = 0$  if and only if  $f \equiv 0$ .

This lemma directly follows from Lemma 3.2 and Lemma 3.4 in [2]. We omit the proof here.

**Remark 3.1** *It is trivial that Lemma 3.1 and Lemma 3.2 are available with  $v = \varepsilon u^0$  if  $f \equiv 0$ . Therefore we have to prove Theorem 2.1 only for (2.17) because all the estimates for (2.12) follow from setting  $f \equiv 0$  which is a special case of (2.17).*

Taking into account of these lemmas, we shall introduce *a priori* estimate in the weighted  $L^\infty$  space as follows. It is obvious that  $v$  is a global classical solution of (2.16). Therefore our unknown function shall be  $U = u - v$ . Then, (2.17) can be rewritten into

$$U = L(F(U + v)). \quad (3.8)$$

In what follows, we shall construct a solution of (3.8) in a weighted  $L^\infty$  space. For this purpose, define the sequence of functions  $\{U_l\}_{l \in \mathbf{N}}$  by

$$U_l = L(F(U_{l-1} + U_0)), \quad U_0 = v \quad \text{and} \quad U_{00} = v_0, \quad U_{01} = v_1, \quad (3.9)$$

where  $L$  is the one in (2.13). We denote a weighted  $L^\infty$  norm of  $U$  by

$$\|U\| = \sup_{(x,t) \in \mathbf{R}^n \times [0,T]} \{w(|x|, t) |U(x, t)|\} \quad (3.10)$$

with a weighted function

$$w(r, t) = \begin{cases} \tau_+(r, t)^{(n-1)/2} \tau_-(r, t)^q & \text{if } p > \frac{n+1}{n-1}, \\ \tau_+(r, t)^{(n-1)/2} \left( \log 4 \frac{\tau_+(r, t)}{\tau_-(r, t)} \right)^{-1} & \text{if } p = \frac{n+1}{n-1}, \\ \tau_+(r, t)^{(n-1)/2+q} & \text{if } 1 < p < \frac{n+1}{n-1}, \end{cases} \quad (3.11)$$

where we set

$$\tau_+(r, t) = \frac{t + r + 2k}{k}, \quad \tau_-(r, t) = \frac{t - r + 2k}{k}$$

and  $q$  is defined by

$$q = \frac{n-1}{2}p - \frac{n+1}{2}. \quad (3.12)$$

In order to get a  $C^1$  solution of (3.8), we shall show the convergence of  $\{U_l\}_{l \in \mathbf{N}}$  in a function space  $X$  defined by

$$X = \{U \in C^1(\mathbf{R}^n \times [0, T]) : \|U\|_X < \infty, \text{supp } U(x, t) \subset \{|x| \leq t + k\}\}$$

which equips a norm

$$\|U\|_X = \sum_{|\alpha| \leq 1} \|\nabla_x^\alpha U\|.$$

In view of (2.17), we note that  $\partial U / \partial t$  can be expressed in terms of  $\nabla_x U$ . Hence we consider spatial derivatives of  $U$  only. Moreover, we see that  $X$  is a Banach space for any fixed  $T > 0$ . Because it follows from the definition of the norm (3.10) that there exists a positive constant  $C_T$  depending on  $T$  such that

$$\|U\| \geq C_T |U(x, t)|, \quad t \in [0, T].$$

Later, we shall make use of Hölder's inequality

$$\| |U_1|^a |U_2|^b \| \leq \|U_1\|^a \|U_2\|^b, \quad a + b = 1, \quad a, b \in [0, 1]. \quad (3.13)$$

Further more, we denote  $\partial / \partial x_i$  by  $\partial_i$  for  $i = 1, 2, \dots, n$ , and set

$$\begin{aligned} \partial^j W_l &= \max_{|\alpha| \leq j} \{ |\nabla_x^\alpha U_l|, |\nabla_x^\alpha U_{l-1}| \}, \\ \partial^j W_0 &= \max_{|\alpha| \leq j} \{ |\nabla_x^\alpha U_0| \}, \quad \partial^j W_{0a} = \max_{|\alpha| \leq j} \{ |\nabla_x^\alpha U_{0a}| \} \end{aligned}$$

for  $j, a = 0, 1$ . One may omit  $\partial^0$  and denote  $\partial = \partial^1$ .

## 4 A priori estimate

In this section, we show *a priori* estimate which plays a key role in the contraction mapping argument. The following lemma is one of the most essential estimates.

**Lemma 4.1** *Let  $L$  be the linear integral operator defined by (2.13). Assume that  $U \in C^0(\mathbf{R}^n \times [0, T])$  with  $\text{supp } U \subset \{(x, t) \in \mathbf{R}^n \times [0, T] : |x| \leq t + k\}$  and  $\|U\| < \infty$ . Then, there exists a positive constant  $C$  independent of  $k$  and  $T$  such that*

$$\|L(|U|^p)\| \leq Ck^2\|U\|^p D(T), \quad (4.1)$$

where  $D(T)$  is defined by

$$D(T) = \begin{cases} 1 & \text{if } p > p_0(n), \\ \log \frac{2T+3k}{k} & \text{if } p = p_0(n), \\ \left(\frac{2T+3k}{k}\right)^{\gamma(p,n)/2} & \text{if } 1 < p < p_0(n) \end{cases} \quad (4.2)$$

and  $\gamma(p, n)$  is the one in (2.2).

We note that Lemma 4.1 is not sufficient to cover all the cases on the exponent. In fact, we need the following variants of *a priori* estimate up to space dimensions.

**Lemma 4.2** *Let  $n = 5, 7, 9, \dots$  and  $L$  be the linear integral operator defined by (2.13). Assume that  $U, U_0 \in C^0(\mathbf{R}^n \times [0, T])$  with  $\text{supp } U \subset \{(x, t) \in \mathbf{R}^n \times [0, T] : |x| \leq t + k\}$ ,  $\text{supp } U_0 \subset \{(x, t) \in \mathbf{R}^n \times [0, T] : t - k \leq |x| \leq t + k\}$  and  $\|U\|, \|\tau_+^{(n-1)/2} U_0 w^{-1}\| < \infty$ . Then, there exists a positive constant  $C_{n,\nu,p}$  depending on  $n, \nu$  and  $p$  such that*

$$\|L(|U_0|^{p-\nu}|U|^\nu)\| \leq C_{n,\nu,p} k^2 \left\| \frac{\tau_+^{(n-1)/2}}{w} U_0 \right\|^{p-\nu} \|U\|^\nu E_\nu(T), \quad (4.3)$$

where  $0 \leq \nu \leq p$ .  $E_\nu(T)$  is defined by

$$E_\nu(T) = \begin{cases} 1 & \text{if } p > \frac{n+1}{n-1}, \\ \left(\frac{2T+3k}{k}\right)^{\nu\delta} & \text{if } p = \frac{n+1}{n-1}, \\ \left(\frac{2T+3k}{k}\right)^{-\nu q} & \text{if } p < \frac{n+1}{n-1} \end{cases} \quad \text{for } 0 \leq \nu < p, \quad (4.4)$$

where  $q$  is the one in (3.12) and  $\delta$  is a small positive constant. When  $\nu = p$ , (4.3) coincides with (4.1) as  $E_p(T) = D(T)$  and  $C_{n,p,p} = C$ .

**Lemma 4.3** *Let  $n = 4, 6, 8, \dots$  and  $L$  be the linear integral operator defined by (2.13). Assume that  $U, U_{00}, U_{01} \in C^0(\mathbf{R}^n \times [0, T])$  with  $\text{supp}(U, U_{00}, U_{01}) \subset$*

$\{(x, t) \in \mathbf{R}^n \times [0, T] : |x| \leq t + k\}$  and  $\|U\|, \|(\tau_+ \tau_-)^{(n-1)/2} U_{0a} (w \tau_-^a)^{-1}\| < \infty$  ( $a = 0, 1$ ). Then, there exists a positive constant  $C_{n,\nu,p}$  depending on  $n$ ,  $\nu$ , and  $p$  such that

$$\|L(|U_{0a}|^{p-\nu} |U|^\nu)\| \leq C_{n,\nu,p} k^2 \left\| (\tau_+ \tau_-)^{(n-1)/2} \frac{U_{0a}}{w \tau_-^a} \right\|^{p-\nu} \|U\|^\nu E_{\nu,a}(T), \quad (4.5)$$

where  $0 \leq \nu \leq p$  and  $a = 0, 1$ . When  $0 \leq \nu < p$ ,  $E_{\nu,a}(T)$  is defined by

$$E_{\nu,a}(T) = \begin{cases} 1 & \text{if } \mu < -1, \\ \log \frac{2T+3k}{k} & \text{if } \mu = -1, \\ \left(\frac{2T+3k}{k}\right)^{1+\mu} & \text{if } \mu > -1 \end{cases} \quad \text{for } p > \frac{n+1}{n-1}, \quad (4.6)$$

where  $\mu = (p - \nu) \left(a - \frac{n-1}{2}\right) - \nu q$  and  $q$  is the one in (3.12), and

$$E_{\nu,a}(T) = \begin{cases} \log \frac{2T+3k}{k} & \text{if } \sigma = -1, \nu = 0, \\ \left(\frac{2T+3k}{k}\right)^{1+\sigma} & \text{if } \sigma > -1, \\ \left(\frac{2T+3k}{k}\right)^{\nu\delta} & \text{otherwise} \end{cases} \quad \text{for } p = \frac{n+1}{n-1}, \quad (4.7)$$

where  $\sigma = (p - \nu) \left(a - \frac{n-1}{2}\right)$  and  $\delta$  stands for any positive constant, and

$$E_{\nu,a}(T) = \begin{cases} \left(\frac{2T+3k}{k}\right)^{-\nu q} & \text{if } \sigma < -1, \\ \log \frac{2T+3k}{k} \left(\frac{2T+3k}{k}\right)^{-\nu q} & \text{if } \sigma = -1, \text{ for } p < \frac{n+1}{n-1}. \\ \left(\frac{2T+3k}{k}\right)^{1+\mu} & \text{if } \sigma > -1 \end{cases} \quad (4.8)$$

When  $\nu = p$ , (4.5) coincides with (4.1) as  $E_{p,a}(T) = D(T)$  for  $a = 0, 1$  and  $C_{n,p,p} = C$ .

**Lemma 4.4** Suppose that the same assumption as in Lemma 4.3 is fulfilled. Then, there exists a positive constant  $C_{n,\nu,p}$  depending on  $n$ ,  $\nu$ , and  $p$  such that

$$\begin{aligned} & \|L(|U_{00}|^{p-\nu} |U_{01}|^\nu)\| \\ & \leq C_{n,\nu,p} k^2 \left\| (\tau_+ \tau_-)^{(n-1)/2} \frac{U_{00}}{w} \right\|^{p-\nu} \left\| (\tau_+ \tau_-)^{(n-1)/2} \frac{U_{01}}{w \tau_-} \right\|^\nu F_\nu(T), \end{aligned} \quad (4.9)$$



where  $0 \leq \nu \leq p$ . When  $0 < \nu < p$ ,  $F_\nu(T)$  is defined by

$$F_\nu(T) = \begin{cases} 1 & \text{if } \kappa < -1, \\ \log \frac{2T+3k}{k} & \text{if } \kappa = -1, \\ \left( \frac{2T+3k}{k} \right)^{1+\kappa} & \text{if } \kappa > -1, \end{cases} \quad (4.10)$$

where  $\kappa = \nu - \frac{n-1}{2}p$ . When  $\nu = 0$  or  $\nu = p$ , (4.9) coincides with (4.5) as  $F_0(T) = E_{0,0}(T)$  for  $a = \nu = 0$  and  $F_p(T) = E_{0,1}(T)$  for  $a = 1$ ,  $\nu = p$ .

Four lemmas above follows from the following basic estimate.

**Lemma 4.5 (Basic estimate)** *Let  $L$  be the linear integral operator defined by (2.13) and  $a_1 \geq 0$ ,  $a_2 \in \mathbf{R}$  and  $a_3 \geq 0$ . Then, there exists a positive constant  $C_{n,p,a_1,a_2,a_3}$  such that*

$$\begin{aligned} & L \left\{ \tau_+^{-(n-1)p/2+a_1} \tau_-^{a_2} (\log(4\tau_+/\tau_-))^{a_3} \right\} (x, t) \\ & \leq C_{n,p,a_1,a_2,a_3} k^2 w(r, t)^{-1} \left( \frac{2T+3k}{k} \right)^{a_1} E_{a_1,a_2,a_3}(T) \end{aligned} \quad (4.11)$$

for  $|x| \leq t + k$ ,  $t \in [0, T]$ , where  $E_{a_1,a_2,a_3}(T)$  is defined by

$$E_{a_1,a_2,a_3}(T) = \begin{cases} 1 & \text{if } a_2 < -1 \text{ and } a_3 = 0, \\ \log \frac{2T+3k}{k} & \text{if } a_2 = -1 \text{ and } a_3 = 0, \\ \left( \frac{2T+3k}{k} \right)^{\delta a_3} & \text{if } a_2 \leq -1 \text{ and } a_3 > 0, \\ \left( \frac{2T+3k}{k} \right)^{1+a_2} & \text{if } a_2 > -1, \end{cases} \quad (4.12)$$

where  $\delta$  stands for any positive constant.

**Proof.** First we employ the following fundamental identity for spherical means.

**Lemma 4.6 (John [13])** *Let  $b \in C([0, \infty))$ . Then, the identity*

$$\int_{|\omega|=1} b(|x + \rho\omega|) dS_\omega = 2^{3-n} \omega_{n-1} (r\rho)^{2-n} \int_{|\rho-r|}^{\rho+r} \lambda h(\lambda, \rho, r) b(\lambda) d\lambda \quad (4.13)$$

holds for  $x \in \mathbf{R}^n$ ,  $r = |x|$  and  $\rho > 0$ , where

$$h(\lambda, \rho, r) = \{\lambda^2 - (\rho - r)^2\}^{(n-3)/2} \{(\rho + r)^2 - \lambda^2\}^{(n-3)/2}. \quad (4.14)$$

See [13] for the proof of this lemma.

In order to continue the proof of Lemma 4.5, we need radially symmetric versions of  $L$  which follows from Lemma 4.6. From now on, a positive constant  $C$  depending only on  $n$  and  $p$  may change from line to line.

**Lemma 4.7** *Let  $L$  be the linear integral operator defined by (2.13) and  $\Psi = \Psi(|x|, t) \in C([0, \infty)^2)$ ,  $x \in \mathbf{R}^n$ . Then,*

$$L(\Psi)(x, t) = L_{\text{odd}}(\Psi)(r, t), \quad r = |x| \quad (4.15)$$

holds for  $n = 5, 7, 9, \dots$  and

$$L(\Psi)(x, t) = L_{\text{even},1}(\Psi)(r, t) + L_{\text{even},2}(\Psi)(r, t), \quad r = |x| \quad (4.16)$$

holds for  $n = 4, 6, 8, \dots$ , where  $L_{\text{odd}}(\Psi)$  is defined by

$$\begin{aligned} L_{\text{odd}}(\Psi)(r, t) \\ = Cr^{2-n} \int_0^t (t-\tau)^{3-n} d\tau \int_{|t-\tau-r|}^{t-\tau+r} \lambda h(\lambda, t-\tau, r) \Psi(\lambda, \tau) d\lambda \end{aligned} \quad (4.17)$$

and each  $L_{\text{even},i}(\Psi)$  ( $i = 1, 2$ ) is defined by

$$\begin{aligned} L_{\text{even},1}(\Psi)(r, t) = & Cr^{2-n} \int_0^t (t-\tau)^{2-n} d\tau \int_{|t-r-\tau|}^{t+r-\tau} \lambda \Psi(\lambda, \tau) d\lambda \times \\ & \times \int_{|\lambda-r|}^{t-\tau} \frac{\rho h(\lambda, \rho, r)}{\sqrt{(t-\tau)^2 - \rho^2}} d\rho, \end{aligned} \quad (4.18)$$

$$\begin{aligned} L_{\text{even},2}(\Psi)(r, t) = & Cr^{2-n} \int_0^{(t-r)_+} (t-\tau)^{2-n} d\tau \int_0^{t-r-\tau} \lambda \Psi(\lambda, \tau) d\lambda \times \\ & \times \int_{|\lambda-r|}^{\lambda_0+r} \frac{\rho h(\lambda, \rho, r)}{\sqrt{(t-\tau)^2 - \rho^2}} d\rho. \end{aligned} \quad (4.19)$$

Here the usual notation  $a_+ = \max\{a, 0\}$  is used.

**Proof.** (4.15) immediately follows from Lemma 4.6. For (4.16), we make use of changing variables by  $y - x = (t - \tau)\xi$  in (2.13). Then, we obtain

$$L(\Psi)(x, t) = C \int_0^t (t-\tau)^{2-n} d\tau \int_{|y-x| \leq t-\tau} \frac{\Psi(|y|, \tau)}{\sqrt{(t-\tau)^2 - |y-x|^2}} dy.$$

Introducing polar coordinates, we have

$$\begin{aligned} L(\Psi)(x, t) = & C \int_0^t (t-\tau)^{2-n} d\tau \int_0^{t-\tau} \frac{\rho^{n-1} d\rho}{\sqrt{(t-\tau)^2 - \rho^2}} \times \\ & \times \int_{|\omega|=1} \Psi(|x + \rho\omega|, \tau) dS_\omega. \end{aligned}$$

Thus Lemma 4.6 yields

$$\begin{aligned} L(\Psi)(x, t) &= Cr^{2-n} \int_0^t (t-\tau)^{2-n} d\tau \int_0^{t-\tau} \frac{\rho d\rho}{\sqrt{(t-\tau)^2 - \rho^2}} \times \\ &\times \int_{|\rho-r|}^{\rho+r} \lambda \Psi(\lambda, \tau) h(\lambda, \rho, r) d\lambda. \end{aligned} \quad (4.20)$$

Therefore, (4.16) follows from inverting the order of  $(\rho, \lambda)$ -integral in (4.20).  
□

In order to estimate the kernel  $h(\lambda, \rho, r)$ , we need the following lemma.

**Lemma 4.8 (Agemi, Kubota and Takamura [2])** *Let  $h(\lambda, \rho, r)$  be the one in (4.14). Suppose that  $|\rho - r| \leq \lambda \leq \rho + r$  and  $\rho \geq 0$ . Then, the inequality*

$$|\lambda - r| \leq \rho \leq \lambda + r \quad (4.21)$$

*holds. Moreover, the following three estimates are available.*

$$h(\lambda, \rho, r) \leq Cr^{n-3} \lambda^{n-3}, \quad (4.22)$$

$$h(\lambda, \rho, r) \leq C\rho^{n-3} r^{(n-3)/2} \lambda^{(n-3)/2}, \quad (4.23)$$

$$h(\lambda, \rho, r) \leq Cr^{n-3} \rho^{n-3}. \quad (4.24)$$

See pp.257-258 in [2] for the proof of this lemma.

Let us continue the proof of Lemma 4.5. For simplicity, we set

$$\begin{aligned} I_{\text{odd}}(r, t) &= L_{\text{odd}} \left\{ \tau_+^{-(n-1)p/2+a_1} \tau_-^{a_2} (\log(4\tau_+/\tau_-))^{a_3} \right\} (r, t), \\ I_{\text{even},i}(r, t) &= L_{\text{even},i} \left\{ \tau_+^{-(n-1)p/2+a_1} \tau_-^{a_2} (\log(4\tau_+/\tau_-))^{a_3} \right\} (r, t) \quad (i = 1, 2). \end{aligned}$$

**Estimates for  $I_{\text{odd}}$  and  $I_{\text{even},1}$ .** We shall estimate  $I_{\text{odd}}$  and  $I_{\text{even},1}$  on the following three domains.

$$\begin{aligned} D_1 &= \{(r, t) \mid r \geq t - r > -k \text{ and } r \geq 2k\}, \\ D_2 &= \{(r, t) \mid r \geq t - r > -k \text{ and } r \leq 2k\}, \\ D_3 &= \{(r, t) \mid t - r \geq r\}. \end{aligned}$$

(i) Estimate in  $D_1$ ,

Making use of (4.23), we get

$$\begin{aligned} I_{\text{odd}}(r, t) &\leq Cr^{-(n-1)/2} \int_0^t d\tau \int_{|t-\tau-r|}^{t+r-\tau} \lambda^{(n-1)/2} \tau_-(\lambda, \tau)^{a_2} \times \\ &\times \tau_+(\lambda, \tau)^{-(n-1)p/2+a_1} \left( \log 4 \frac{\tau_+(\lambda, \tau)}{\tau_-(\lambda, \tau)} \right)^{a_3} d\lambda \end{aligned} \quad (4.25)$$

and

$$\begin{aligned}
I_{even,1}(r, t) &\leq Cr^{-(n-1)/2} \int_0^t (t-\tau)^{2-n} d\tau \int_{|t-\tau-r|}^{t+r-\tau} \lambda^{(n-1)/2} \times \\
&\quad \times \tau_+(\lambda, \tau)^{-(n-1)p/2+a_1} \tau_-(\lambda, \tau)^{a_2} \times \\
&\quad \times \left( \log 4 \frac{\tau_+(\lambda, \tau)}{\tau_-(\lambda, \tau)} \right)^{a_3} d\lambda \int_{|\lambda-r|}^{t-\tau} \frac{\rho^{n-2}}{\sqrt{(t-\tau)^2 - \rho^2}} d\rho.
\end{aligned} \tag{4.26}$$

If one apply the simple inequality

$$\int_{|\lambda-r|}^{t-\tau} \frac{\rho^{n-2}}{\sqrt{(t-\tau)^2 - \rho^2}} d\rho \leq (t-\tau)^{n-2} \quad \text{for } 0 \leq \tau \leq t \tag{4.27}$$

to the right-hand side of (4.26), the same quantity as the right-hand side of (4.25) appears. Hence, we shall estimate for  $I_{odd}$  only from now on.

Changing variables in (4.25) by

$$\alpha = \tau + \lambda, \quad \beta = \tau - \lambda, \tag{4.28}$$

we get

$$\begin{aligned}
I_{odd}(r, t) &\leq Cr^{-(n-1)/2} \int_{-k}^{t-r} \left( \frac{\beta + 2k}{k} \right)^{a_2} d\beta \int_{|t-r|}^{t+r} (\alpha - \beta)^{(n-1)/2} \times \\
&\quad \times \left( \frac{\alpha + 2k}{k} \right)^{-(n-1)p/2+a_1} \left( \log 4 \frac{\alpha + 2k}{\beta + 2k} \right)^{a_3} d\alpha.
\end{aligned}$$

It follows from

$$\frac{r}{k} = \frac{2r + r + r}{4k} \geq \frac{\tau_+(r, t)}{4}$$

that

$$\begin{aligned}
&I_{odd}(r, t) \\
&\leq C\tau_+(r, t)^{-(n-1)/2} \left( \frac{t+r+2k}{k} \right)^{a_1} \int_{-k}^{t-r} \left( \frac{\beta + 2k}{k} \right)^{a_2} d\beta \times \\
&\quad \times \int_{t-r}^{t+r} \left( \frac{\alpha + 2k}{k} \right)^{-1-q} \left( \log 4 \frac{\alpha + 2k}{\beta + 2k} \right)^{a_3} d\alpha.
\end{aligned} \tag{4.29}$$

When  $a_3 = 0$ ,  $\alpha$ -integral in (4.29) is dominated by

$$\begin{cases} Ck\tau_-^{-q} & \text{if } p > \frac{n+1}{n-1}, \\ k \log \frac{\tau_+}{\tau_-} & \text{if } p = \frac{n+1}{n-1}, \\ Ck\tau_+^{-q} & \text{if } p < \frac{n+1}{n-1} \end{cases}$$

and  $\beta$ -integral oin (4.29) is dominated by

$$\begin{cases} \frac{k}{-(1+a_2)} & \text{if } a_2 < -1, \\ k \log \frac{t-r+2k}{k} & \text{if } a_2 = -1, \\ \frac{k}{(1+a_2)} \left( \frac{t-r+2k}{k} \right)^{1+a_2} & \text{if } a_2 > -1. \end{cases}$$

(4.11) is now established for  $a_3 = 0$ .

When  $a_3 > 0$ , we employ the following simple lemma.

**Lemma 4.9** *Let  $\delta > 0$  be any given constant. Then, we have*

$$\log X \leq \frac{X^\delta}{\delta} \text{ for } X \geq 1. \quad (4.30)$$

The proof of this lemma follows from elementary computation. We shall omit it. Then, it follows from Lemma 4.9 that

$$\begin{aligned} I_{odd}(r, t) &\leq C(4\delta^{-1})^{a_3} \tau_+(r, t)^{-(n-1)/2} \left( \frac{t+r+2k}{k} \right)^{a_1+\delta a_3} \times \\ &\quad \times \int_{-k}^{t-r} \left( \frac{\beta+2k}{k} \right)^{a_2-\delta a_3} d\beta \int_{t-r}^{t+r} \left( \frac{\alpha+2k}{k} \right)^{-1-q} d\alpha. \end{aligned}$$

The  $\alpha$ -integral above can be estimated by the same manner in the case of  $a_3 = 0$ . The  $\beta$ -integral is dominated by

$$\begin{cases} \frac{-k}{1+a_2-\delta a_3} & \text{if } a_2 \leq -1, \\ \frac{k}{1+a_2-\delta a_3} \left( \frac{t-r+2k}{k} \right)^{1+a_2-\delta a_3} & \text{if } a_2 > -1 \end{cases} \quad (4.31)$$

with  $\delta > 0$  satisfying  $1+a_2-\delta a_3 > 0$ . Therefore  $I_{odd}$  and  $I_{even,1}$  are bounded in  $D_1$  by the quantity in the right-hand side of (4.11) as desired.

(ii) Estimate in  $D_2$ .

By making use of (4.24), we have

$$\begin{aligned} I_{odd}(r, t) &\leq C r^{-1} \int_0^t d\tau \int_{|t-\tau-r|}^{t+r-\tau} \lambda \tau_+(\lambda, \tau)^{-(n-1)p/2+a_1} \times \\ &\quad \times \tau_-(\lambda, \tau)^{a_2} \left( \log 4 \frac{\tau_+(\lambda, \tau)}{\tau_-(\lambda, \tau)} \right)^{a_3} d\lambda \end{aligned} \quad (4.32)$$

and

$$I_{even,1}(r, t) \leq Cr^{-1} \int_0^t (t-\tau)^{-1} d\tau \int_{|t-\tau-r|}^{t+r-\tau} \lambda \tau_+(\lambda, \tau)^{-(n-1)p/2+a_1} \times \\ \times \tau_-(\lambda, \tau)^{a_2} \left( \log 4 \frac{\tau_+(\lambda, \tau)}{\tau_-(\lambda, \tau)} \right)^{a_3} d\lambda \int_{|\lambda-r|}^{t-\tau} \frac{\rho}{\sqrt{(t-\tau)^2 - \rho^2}} d\rho.$$

Similarly to the estimate in  $D_1$ , the simple inequality

$$\int_{|\lambda-r|}^{t-\tau} \frac{\rho}{\sqrt{(t-\tau)^2 - \rho^2}} d\rho \leq t-\tau \quad \text{for } 0 \leq \tau \leq t$$

helps us to estimate  $I_{odd}$  only. Changing variables by (4.28), we get

$$I_{odd}(r, t) \leq Ckr^{-1} \left( \frac{t+r+2k}{k} \right)^{a_1} \int_{-k}^{t-r} \left( \frac{\beta+2k}{k} \right)^{a_2} d\beta \times \\ \times \int_{t-r}^{t+r} \left( \frac{\alpha+2k}{k} \right)^{1-(n-1)p/2} \left( \log 4 \frac{\alpha+2k}{\beta+2k} \right)^{a_3} d\alpha. \quad (4.33)$$

Note that both  $\tau_+$  and  $\tau_-$  are numerical constants in this domain, and that the integrand of both  $\alpha$ -integral and  $\beta$ -integral in (4.33) is numerical constant  $C_{a_1, a_2, a_3}$  depending on  $a_1$ ,  $a_2$  and  $a_3$ . Hence we have

$$I_{odd}(r, t) \leq CC_{a_1, a_2, a_3} kr^{-1} \int_{-k}^{t-r} d\beta \int_{t-r}^{t+r} d\alpha \leq CC_{a_1, a_2, a_3} k^2. \quad (4.34)$$

This is the desired estimate in  $D_2$ .

(iii) Estimate in  $D_3$ .

By the same reason, we have to estimate  $I_{odd}$  in (4.33) only. In  $D_3$ , since  $1 - (n-1)p/2 < 0$  is trivial, we get

$$\left( \frac{\alpha+2k}{k} \right)^{1-(n-1)p/2} \leq \left( \frac{t-r+2k}{k} \right)^{1-(n-1)p/2} \leq Cw(r, t)^{-1}$$

because  $t-r \geq r$  is equivalent to  $3(t-r) \geq t+r$ . Hence, when  $a_3 = 0$ , we obtain

$$I_{odd}(r, t) \leq Ckw(r, t)^{-1} \left( \frac{t+r+2k}{k} \right)^{a_1} \int_{-k}^{t-r} \left( \frac{\beta+2k}{k} \right)^{a_2} d\beta.$$

When  $a_3 > 0$ , due to Lemma 4.9, we have

$$I_{odd}(r, t) \leq C(4\delta^{-1})^{a_3} kw(r, t)^{-1} \times \\ \times \left( \frac{t+r+2k}{k} \right)^{a_1+\delta a_3} \int_{-k}^{t-r} \left( \frac{\beta+2k}{k} \right)^{a_2-\delta a_3} d\beta.$$

Therefore, in view of (4.31),  $I_{odd}$  and  $I_{even,1}$  are bounded in  $D_3$  by the quantity in the right-hand side of (4.11).

**Estimates for  $I_{even,2}$ .** We shall estimate  $I_{even,2}$  on the following three domains.

$$\begin{aligned} D_4 &= \{(r, t) \mid 0 < t - r \leq k \text{ and } t \leq 2k\}, \\ D_5 &= \{(r, t) \mid 0 < t - r \leq k \text{ and } t \geq 2k\}, \\ D_6 &= \{(r, t) \mid t - r \geq k\}. \end{aligned}$$

(iv) Estimate in  $D_4$ ,

Note that  $\tau_+$  and  $\tau_-$  are numerical constants in this case. By virtue of (4.21) and (4.22), we get

$$\begin{aligned} I_{even,2}(r, t) &\leq CC_{a_1, a_2, a_3} r^{-1} \int_0^{t-r} (t - \tau)^{2-n} d\tau \int_0^{t-r-\tau} \lambda^{n-2} d\lambda \\ &\quad \times \int_{|\lambda-r|}^{\lambda+r} \frac{\rho}{\sqrt{(t - \tau)^2 - \rho^2}} d\rho. \end{aligned}$$

It follows from

$$\int_{|\lambda-r|}^{\lambda+r} \frac{\rho d\rho}{\sqrt{(t - \tau)^2 - \rho^2}} \leq \frac{2r\lambda}{\sqrt{t - \tau + \lambda + r} \sqrt{t - \tau - \lambda - r}} \quad (4.35)$$

that

$$\begin{aligned} I_{even,2}(r, t) &\leq CC_{a_1, a_2, a_3} \int_0^{t-r} (t - \tau)^{3/2-n} d\tau \int_0^{t-r-\tau} \frac{\lambda^{n-1}}{\sqrt{t - \tau - \lambda - r}} d\lambda. \end{aligned}$$

Noticing that

$$\lambda \leq t - r - \tau \leq t - \tau \quad \text{for } \tau \geq 0,$$

we obtain

$$I_{even,2}(r, t) = CC_{a_1, a_2, a_3} \int_0^{t-r} (t - \tau)^{1/2} d\tau \int_0^{t-r-\tau} \frac{d\lambda}{\sqrt{t - \tau - \lambda - r}}.$$

Making use of (4.28), we have

$$\begin{aligned} I_{even,2}(r, t) &\leq CC_{a_1, a_2, a_3} k^{1/2} \int_{-k}^{t-r} d\beta \int_{\beta}^{t-r} \frac{d\alpha}{\sqrt{t - r - \alpha}} \\ &\leq CC_{a_1, a_2, a_3} k^2. \end{aligned}$$

This is the desired estimate in  $D_4$ .

(v) Estimate in  $D_5$ .

In this domain, (4.11) follows from

$$I_{even,2}(r, t) \leq Ck^2 C_{a_1,1_2,a_3} \tau_+(r, t)^{-(n-1)/2}.$$

To see this, we shall employ (4.21) and (4.23). Then we have

$$\begin{aligned} I_{even,2}(r, t) &\leq Cr^{-(n-1)/2} \int_0^{t-r} (t-\tau)^{2-n} d\tau \int_0^{t-r-\tau} \lambda^{(n-1)/2} \times \\ &\quad \times \tau_+(\lambda, \tau)^{-(n-1)p/2+a_1} \tau_-(\lambda, \tau)^{a_2} \times \\ &\quad \times \left( \log 4 \frac{\tau_+(\lambda, \tau)}{\tau_-(\lambda, \tau)} \right)^{a_3} d\lambda \int_{|\lambda-r|}^{\lambda+r} \frac{\rho^{n-2}}{\sqrt{(t-\tau)^2 - \rho^2}} d\rho. \end{aligned}$$

Similarly to (4.34), it follows from (4.27) that

$$I_{even,2}(r, t) \leq CC_{a_1,a_2,a_3} r^{-(n-1)/2} k^2.$$

In  $D_5$ , we have  $r \geq k$  which implies  $r/k \geq C\tau_+(r, t)$ . Hence, we obtain the desired estimate.

(vi) Estimate in  $D_6$ .

By virtue of (4.22) and (4.35), we get

$$\begin{aligned} I_{even,2}(r, t) &\leq C \int_0^{t-r} (t-\tau)^{3/2-n} d\tau \int_0^{t-r-\tau} \frac{\lambda^{n-1}}{\sqrt{t-\tau-\lambda-r}} \times \\ &\quad \times \tau_+(\lambda, \tau)^{-(n-1)p/2+a_1} \tau_-(\lambda, \tau)^{a_2} \left( \log 4 \frac{\tau_+(\lambda, \tau)}{\tau_-(\lambda, \tau)} \right)^{a_3} d\lambda. \end{aligned}$$

Then, we divide the integral of the right-hand side above as

$$I_{even,3}(r, t) + I_{even,4}(r, t),$$

where

$$\begin{aligned} I_{even,3}(r, t) &= C \int_0^{(t-r)/2} (t-\tau)^{3/2-n} d\tau \int_0^{t-r-\tau} \frac{\lambda^{n-1}}{\sqrt{t-\tau-\lambda-r}} \times \\ &\quad \times \tau_+(\lambda, \tau)^{-(n-1)p/2+a_1} \tau_-(\lambda, \tau)^{a_2} \left( \log 4 \frac{\tau_+(\lambda, \tau)}{\tau_-(\lambda, \tau)} \right)^{a_3} d\lambda \end{aligned}$$

and

$$\begin{aligned} I_{even,4}(r, t) &= C \int_{(t-r)/2}^{t-r} (t-\tau)^{3/2-n} d\tau \int_0^{t-r-\tau} \frac{\lambda^{n-1}}{\sqrt{t-\tau-\lambda-r}} \times \\ &\quad \times \tau_+(\lambda, \tau)^{-(n-1)p/2+a_1} \tau_-(\lambda, \tau)^{a_2} \left( \log 4 \frac{\tau_+(\lambda, \tau)}{\tau_-(\lambda, \tau)} \right)^{a_3} d\lambda. \end{aligned}$$



First, we shall estimate  $I_{even,3}$ . It follows from (4.28) that

$$\begin{aligned} & I_{even,3}(r, t) \\ & \leq Ck^{1/2} \left( \frac{t+r+2k}{k} \right)^{3/2-n} \left( \frac{t-r+2k}{k} \right)^{n-1-(n-1)p/2+a_1} \times \\ & \quad \times \int_{-k}^{t-r} \left( \frac{\beta+2k}{k} \right)^{a_2} d\beta \int_{\beta}^{t-r} \left( \log 4 \frac{\alpha+2k}{\beta+2k} \right)^{a_3} \frac{d\alpha}{\sqrt{t-r-\alpha}} \end{aligned}$$

because of  $n-1-(n-1)p/2 \geq 0$  for  $p \leq 2$ . When  $a_3 = 0$ , we get

$$\begin{aligned} I_{even,3}(r, t) & \leq Ck \left( \frac{t+r+2k}{k} \right)^{3/2-n} \times \\ & \quad \times \left( \frac{t-r+2k}{k} \right)^{-q+(n-2)/2+a_1} \int_{-k}^{t-r} \left( \frac{\beta+2k}{k} \right)^{a_2} d\beta. \end{aligned}$$

Hence the desired estimate follows from

$$\frac{\tau_-(r, t)^{-q+(n-2)/2}}{\tau_+(r, t)^{n-3/2}} \leq Cw(r, t)^{-1} \quad (4.36)$$

in this case. When  $a_3 > 0$ , Lemma 4.9 yields that

$$\begin{aligned} I_{even,3}(r, t) & \leq Ck(4\delta^{-1})^{a_3} \left( \frac{t+r+2k}{k} \right)^{3/2-n} \times \\ & \quad \left( \frac{t-r+2k}{k} \right)^{-q+(n-2)/2+a_1+\delta a_3} \int_{-k}^{t-r} \left( \frac{\beta+2k}{k} \right)^{a_2-\delta a_3} d\beta. \end{aligned}$$

Hence the desired estimate follows from (4.31) and (4.36).

Next, we shall estimate  $I_{even,4}$ . If  $r \geq t-r \geq k$ , (4.28) yields that

$$\begin{aligned} I_{even,4}(r, t) & \leq \frac{Ck^{n-1}}{r^{n-3/2}} \left( \frac{t-r+2k}{k} \right)^{n-1-(n-1)p/2+a_1} \times \\ & \quad \times \int_{-k}^{t-r} \left( \frac{\beta+2k}{k} \right)^{a_2} d\beta \int_{\beta}^{t-r} \frac{d\alpha}{\sqrt{t-r-\alpha}} \left( \log 4 \frac{\alpha+2k}{\beta+2k} \right)^{a_3}. \end{aligned}$$

In this case, we have  $r/k \geq C\tau_+(r, t)$ , so that (4.11) follows from the same argument as for  $I_{even,3}$ . On the other hand, if  $t-r \geq r$  and  $t-r \geq k$ , we have

$$\tau + \lambda + 2k \geq \frac{t-r}{2} + 2k \geq \frac{t+r+2k}{6} \text{ for } \tau \geq \frac{t-r}{2}, \lambda \geq 0.$$

Hence (4.35) yields that

$$I_{even,4}(r, t) \leq Ck^{1/2} \left( \frac{t+r+2k}{k} \right)^{1/2-(n-1)p/2+a_1} \times \\ \times \int_0^{t-r} d\tau \int_0^{t-r-\tau} \frac{\tau_-(\lambda, \tau)^{a_2}}{\sqrt{t-\tau-\lambda-r}} \left( \log 4 \frac{\tau_+(\lambda, \tau)}{\tau_-(\lambda, \tau)} \right)^{a_3} d\lambda.$$

Changing variables by (4.28), we have

$$I_{even,4}(r, t) \leq Ck^{1/2} \left( \frac{t+r+2k}{k} \right)^{1/2-(n-1)p/2+a_1} \times \\ \times \int_{-k}^{t-r} \left( \frac{\beta+2k}{k} \right)^{a_2} d\beta \int_{\beta}^{t-r} \left( \log 4 \frac{\alpha+2k}{\beta+2k} \right)^{a_3} \frac{d\alpha}{\sqrt{t-r-\alpha}}.$$

Therefore, applying the simple inequality

$$\tau_+(r, t)^{1-(n-1)p/2} \leq Cw(r, t)^{-1},$$

we obtain the desired estimates by the same argument as for  $I_{even,3}$ . The proof of Lemma 4.5 is now completed.  $\square$

**Proof of Lemma 4.1.** Since (2.13) and (3.10) yield that

$$L(|U|^p)(x, t) \leq \|U\|^p L(w^{-p})(x, t),$$

it is enough to show the inequality

$$w(r, t)L(w^{-p})(x, t) \leq Ck^2 D(T).$$

This is established by (4.11) with setting

$$\begin{cases} a_1 = a_3 = 0, \ a_2 = -pq & \text{if } p > \frac{n+1}{n-1}, \\ a_1 = a_2 = 0, \ a_3 = p & \text{if } p = \frac{n+1}{n-1}, \\ a_1 = -pq, \ a_2 = a_3 = 0 & \text{if } p < \frac{n+1}{n-1}. \end{cases}$$

$\square$

**Proof of Lemma 4.2.** Due to Huygens' principle in Lemma 3.1, one can replace  $\tau_-$  by  $\tau_- \chi_{\{-k \leq t-r \leq k\}}$  in (4.11). Then, the integral with respect to the variable of  $\beta = \tau - \lambda$  is bounded. In order to establish (4.3), it is enough to show the inequality

$$w(r, t)L \left( \tau_+^{-(n-1)(p-\nu)/2} w^{-\nu} \chi_{\{-k \leq t-r \leq k\}} \right) \leq C_{n,\nu,p} k^2 E_\nu(T).$$

This is established by (4.11) with setting

$$\begin{cases} a_1 = a_3 = 0, \ a_2 = -\nu q & \text{if } p > \frac{n+1}{n-1}, \\ a_1 = a_2 = 0, \ a_3 = \nu & \text{if } p = \frac{n+1}{n-1}, \\ a_1 = -\nu q, \ a_2 = a_3 = 0 & \text{if } p < \frac{n+1}{n-1}. \end{cases}$$

□

**Proof of Lemma 4.3 and Lemma 4.4.** In order to prove (4.5) and (4.9), it is enough to show inequalities

$$w(r, t) L \left( \tau_+^{-(n-1)(p-\nu)/2} \tau_-^\sigma w^{-\nu} \right) \leq C_{n,\nu,p} k^2 E_{\nu,a}(T)$$

and

$$w(r, t) L \left( \tau_+^{-(n-1)p/2} \tau_-^\kappa \right) \leq C_{n,\nu,p} k^2 F_\nu(T).$$

If we set

$$\begin{cases} a_1 = a_3 = 0, \ a_2 = \mu & \text{if } p > \frac{n+1}{n-1}, \\ a_1 = 0, \ a_2 = \sigma, \ a_3 = \nu & \text{if } p = \frac{n+1}{n-1}, \\ a_1 = -\nu q, \ a_2 = \sigma, \ a_3 = 0 & \text{if } p < \frac{n+1}{n-1} \end{cases}$$

in (4.11), we have (4.5). If we set  $a_1 = a_3 = 0, \ a_2 = \kappa$  in (4.11), we have (4.9). □

## 5 Lower bound in odd space dimensions

In this section, we prove Theorem 2.1 in odd space dimensions. It is obviously enough for this to show the following proposition.

**Proposition 5.1** *Let  $n = 5, 7, 9, \dots$ . Assume (2.19) and (2.20). Then, there exists a positive constant  $\varepsilon_0 = \varepsilon_0(f, g, n, p, k)$  such that each of (2.12) and (2.17) admits a unique solution  $u \in C^1(\mathbf{R}^n \times [0, T])$  as far as  $T$  satisfies*

$$T \leq \begin{cases} c\varepsilon^{-2p(p-1)/\gamma(p,n)} & \text{if } 1 < p < p_0(n) \\ \exp(c\varepsilon^{-p(p-1)}) & \text{if } p = p_0(n) \end{cases} \quad (5.1)$$

for  $0 < \varepsilon \leq \varepsilon_0$ , where  $c$  is a positive constant independent of  $\varepsilon$ .

Our purpose is to construct a solution of the integral equation (3.8) as a limit of  $\{U_l\}_{l \in \mathbf{N}}$  in  $X$ . To end this, we define a closed subspace  $Y_0$  in  $X$  by

$$Y_0 = \{U \in X : \|\nabla_x^\alpha U\| \leq 2M_0 \varepsilon^p \ (|\alpha| \leq 1)\},$$

where we set

$$M_0 = 2^p p A C_{n,0,p} k^2 C_0^p > 0.$$

Recall that  $A$  is the one in (2.19) and  $C_{n,0,p}$  is the one in (4.3). We note that there exists a positive constant  $C_0$  independent of  $\varepsilon$  which satisfies that

$$\left\| \frac{\tau_+^{(n-1)/2}}{w} \partial W_0 \right\| \leq C_0 \varepsilon \quad \text{for } 0 < \varepsilon \leq 1. \quad (5.2)$$

It is easy to check that this fact follows from the definition of the weight function in (3.11) and the decay estimate for  $U_0 = v$  in (3.4) and (3.5).

**Proof of Proposition 5.1.** First of all, we assume that

$$0 < \varepsilon \leq 1 \quad (5.3)$$

to make use of (5.2). We shall show the convergence of  $\{U_l\}_{l \in \mathbf{N}}$ . The boundedness of  $\{U_l\}_{l \in \mathbf{N}}$ ,

$$\|U_l\| \leq 2M_0 \varepsilon^p \quad (l \in \mathbf{N}), \quad (5.4)$$

can be obtained by induction with respect to  $l$  as follows. Recall that  $L$  is a positive and linear operator by its definition, (2.13). It follows from (2.19), (3.9), (4.3) with  $\nu = 0$  and (5.2) that

$$\begin{aligned} \|U_1\| &\leq \|L\{|F(2U_0)|\}\| \leq 2^p A \|L(|U_0|^p)\| \\ &\leq 2^p A C_{n,0,p} k^2 \|\tau_+^{(n-1)/2} w^{-1} U_0\|^p E_0(T) \leq M_0 \varepsilon^p, \end{aligned} \quad (5.5)$$

where we have used  $E_0(T) = 1$  for  $p > 1$ . Assume that  $\|U_{l-1}\| \leq 2M_0 \varepsilon^p$  ( $l \geq 2$ ). It follows from the simple estimate

$$\begin{aligned} |U_l| &\leq AL(|U_{l-1}| + |U_0|^p) \\ &\leq 2^p A \{L(|U_{l-1}|^p) + L(|U_0|^p)\}, \end{aligned} \quad (5.6)$$

(4.1) and (4.3) with  $\nu = 0$  that

$$\|U_l\| \leq 2^p A k^2 \{C \|U_{l-1}\|^p D(T) + C_{n,0,p} \|\tau_+^{(n-1)/2} w^{-1} U_0\|^p E_0(T)\}.$$

Hence (5.2) and the assumption of the induction yield that

$$\|U_l\| \leq 2^p A C k^2 (2M_0 \varepsilon^p)^p D(T) + M_0 \varepsilon^p.$$

This inequality shows (5.4) provided

$$2^p A C k^2 (2M_0)^p \varepsilon^{p^2} D(T) \leq M_0 \varepsilon^p. \quad (5.7)$$

Next we shall estimate the differences of  $\{U_l\}_{l \in \mathbf{N}}$  under the conditions, (5.3) and (5.7) ensuring the boundedness (5.4). By virtue of (2.19), there exists a  $\theta \in (0, 1)$  such that

$$\begin{aligned} |U_{l+1} - U_l| &= |L\{F(U_l + U_0) - F(U_{l-1} + U_0)\}| \\ &= |L\{F'(U_{l-1} + U_0 + \theta(U_l - U_{l-1}))(U_l - U_{l-1})\}| \\ &\leq p A L \{|U_{l-1} + U_0 + \theta(U_l - U_{l-1})|^{p-1} |U_l - U_{l-1}|\}. \end{aligned} \quad (5.8)$$

Hence we have

$$|U_{l+1} - U_l| \leq 2^{p-1} p A L \{(|3W_l|^{p-1} + |U_0|^{p-1}) |U_l - U_{l-1}|\}. \quad (5.9)$$

Hölder's inequality (3.13) and *a priori* estimate (4.1) yield that

$$\begin{aligned} &\|L(|3W_l|^{p-1} |U_l - U_{l-1}|)\| \\ &= \|L\{(|3W_l|^{(p-1)/p} |U_l - U_{l-1}|^{1/p})^p\}\| \\ &\leq C k^2 \| |3W_l|^{(p-1)/p} |U_l - U_{l-1}|^{1/p} \|^p D(T) \\ &\leq C k^2 \|3W_l\|^{p-1} D(T) \|U_l - U_{l-1}\|. \end{aligned} \quad (5.10)$$

We note that (5.4) implies  $\|W_l\| \leq 2M_0 \varepsilon^p$  ( $l \in \mathbf{N}$ ). Moreover, (4.3) with  $\nu = 1$  implies that

$$\begin{aligned} &\|L(|U_0|^{p-1} |U_l - U_{l-1}|)\| \\ &\leq C_{n,1,p} k^2 \|\tau_+^{(n-1)/2} w^{-1} U_0\|^{p-1} \|U_l - U_{l-1}\| E_1(T). \end{aligned} \quad (5.11)$$

Since (5.4) implies that  $\|W_l\| \leq 2M_0 \varepsilon^p$  for  $l \geq 2$ , the convergence of  $\{U_l\}_{l \in \mathbf{N}}$  follows from

$$\|U_{l+1} - U_l\| \leq \frac{1}{2} \|U_l - U_{l-1}\| \quad \text{for } l \geq 2$$

provided

$$2^{p-1} p A k^2 \{C(6M_0 \varepsilon^p)^{p-1} D(T) + C_{n,1,p} (C_0 \varepsilon)^{p-1} E_1(T)\} \leq \frac{1}{2}, \quad (5.12)$$

In fact, we obtain

$$\|U_{l+1} - U_l\| \leq \frac{1}{2^{l-1}} \|U_2 - U_1\| \quad \text{for } l \geq 2 \quad (5.13)$$

which implies the convergence of  $\{U_l\}_{l \in \mathbf{N}}$ .

Now we shall show the convergence of  $\{\partial_i U_l\}_{l \in \mathbf{N}}$  for  $i = 1, 2, \dots, n$  under the conditions, (5.3), (5.7) and (5.12) which ensure the convergence of  $\{U_l\}_{l \in \mathbf{N}}$ . As before, the boundedness of  $\{\partial_i U_l\}_{l \in \mathbf{N}}$  for  $i = 1, 2, \dots, n$ ,

$$\|\partial_i U_l\| \leq 2M_0 \varepsilon^p \quad (l \in \mathbf{N}, i = 1, 2, \dots, n), \quad (5.14)$$

can be obtained by induction as follows. Similarly to (5.5), we have

$$\begin{aligned} \|\partial_i U_1\| &\leq \|L(|F'(2U_0)2\partial_i U_0|)\| \leq 2^p p A \|L(|U_0|^{p-1} |\partial_i U_0|)\| \\ &\leq 2^p p A \|L(|\partial W_0|^p)\| \leq 2^p p A C_{n,0,p} k^2 \|\tau_+^{(n-1)/2} w^{-1} \partial W_0\|^p E_0(T). \end{aligned}$$

Hence (5.2) and  $E_0(T) = 1$  for  $p > 1$  implies  $\|\partial_i U_1\| \leq M_0 \varepsilon^p$ . Assume that  $\|\partial_i U_{l-1}\| \leq 2M_0 \varepsilon^p$  ( $l \geq 2$ ). We note that this means  $\|\partial W_{l-1}\| \leq 2M_0 \varepsilon^p$  ( $l \geq 2$ ). It follows from (2.19) and (3.9) that

$$\begin{aligned} |\partial_i U_l| &\leq |L(|F'(U_{l-1} + U_0)| |\partial_i (U_{l-1} + U_0)|)| \\ &\leq p A |L(|U_{l-1} + U_0|^{p-1} |\partial_i (U_{l-1} + U_0)|)| \\ &\leq 2^{p-1} p A L\{(|U_{l-1}|^{p-1} + |U_0|^{p-1})(|\partial_i U_{l-1}| + |\partial_i U_0|)\}. \end{aligned}$$

Similarly to (5.10) and (5.11), we obtain that

$$\begin{aligned} \|L(|U_{l-1}|^{p-1} |\partial_i U_{l-1}|)\| &\leq C k^2 \|U_{l-1}\|^{p-1} \|\partial_i U_{l-1}\| D(T), \\ \|L(|U_{l-1}|^{p-1} |\partial_i U_0|)\| &\leq C_{n,p-1,p} k^2 \|U_{l-1}\|^{p-1} \|\tau_+^{(n-1)/2} w^{-1} \partial_i U_0\| E_{p-1}(T), \\ \|L(|U_0|^{p-1} |\partial_i U_{l-1}|)\| &\leq C_{n,1,p} k^2 \|\tau_+^{(n-1)/2} U_0 w^{-1}\|^{p-1} \|\partial_i U_{l-1}\| E_1(T), \\ \|L(|U_0|^{p-1} |\partial_i U_0|)\| &\leq C_{n,0,p} k^2 \|\tau_+^{(n-1)/2} \partial W_0 w^{-1}\|^p E_0(T). \end{aligned}$$

Hence, we get

$$\begin{aligned} \|\partial_i U_l\| &\leq 2^{p-1} p A k^2 \{C \|\partial W_{l-1}\|^p D(T) \\ &\quad + C_{n,p-1,p} \|\partial W_{l-1}\|^{p-1} \|\tau_+^{(n-1)/2} w^{-1} \partial W_0\| E_{p-1}(T) \\ &\quad + C_{n,1,p} \|\partial W_{l-1}\| \|\tau_+^{(n-1)/2} w^{-1} \partial W_0\|^{p-1} E_1(T) \\ &\quad + C_{n,0,p} \|\tau_+^{(n-1)/2} w^{-1} \partial W_0\|^p\} \\ &\leq 2^{p-1} p A k^2 \{C (2M_0 \varepsilon^p)^p D(T) + C_{n,p-1,p} (2M_0 \varepsilon^p)^{p-1} C_0 \varepsilon E_{p-1}(T) \\ &\quad + C_{n,1,p} (2M_0 \varepsilon^p) (C_0 \varepsilon)^{p-1} E_1(T) + C_{n,0,p} (C_0 \varepsilon)^p\}. \end{aligned}$$

This inequality shows (5.14) provided

$$\begin{aligned} (3/2) M_0 \varepsilon^p &\geq 2^{p-1} p A k^2 \{C (2M_0)^p \varepsilon^{p^2} D(T) \\ &\quad + C_{n,p-1,p} (2M_0)^{p-1} C_0 \varepsilon^{p^2-p+1} E_{p-1}(T) \\ &\quad + C_{n,1,p} 2M_0 C_0^{p-1} \varepsilon^{2p-1} E_1(T)\}. \end{aligned} \quad (5.15)$$

Next we shall estimate the differences of  $\{\partial_i U_l\}_{l \in \mathbf{N}}$  for  $i = 1, 2, \dots, n$ , under the conditions, (5.3), (5.7), (5.12) and (5.15) which ensure the convergence of  $\{U_l\}_{l \in \mathbf{N}}$  and the boundedness of  $\{\partial W_l\}_{l \in \mathbf{N}}$ . (3.9) implies that

$$\begin{aligned} & |\partial_i U_{l+1} - \partial_i U_l| \\ &= |L\{F'(U_l + U_0)(\partial_i U_l + \partial_i U_0) - F'(U_{l-1} + U_0)(\partial_i U_{l-1} + \partial_i U_0)\}| \\ &\leq L\{|F'(U_l + U_0)| |\partial_i U_l - \partial_i U_{l-1}|\} \\ &\quad + L\{|F'(U_l + U_0) - F'(U_{l-1} + U_0)| |\partial_i U_{l-1} + \partial_i U_0|\}. \end{aligned}$$

Hence it follows from (2.19) that

$$\begin{aligned} |\partial_i U_{l+1} - \partial_i U_l| &\leq pAL\{|U_l + U_0|^{p-1} |\partial_i U_l - \partial_i U_{l-1}|\} \\ &\quad + pAL\{|U_l - U_{l-1}|^{p-1} |\partial_i U_{l-1} + \partial_i U_0|\} \\ &\leq 2^{p-1} pAL\{(|U_l|^{p-1} + |U_0|^{p-1}) |\partial_i U_l - \partial_i U_{l-1}|\} \\ &\quad + pAL\{|U_l - U_{l-1}|^{p-1} (|\partial_i U_{l-1}| + |\partial_i U_0|)\}. \end{aligned}$$

Similarly to the proof of the convergence of  $\{U_l\}_{l \in \mathbf{N}}$ , we obtain that

$$\begin{aligned} & \|L\{|U_l|^{p-1} |\partial_i U_l - \partial_i U_{l-1}|\}\| \leq Ck^2 \|U_l\|^{p-1} \|\partial_i U_l - \partial_i U_{l-1}\| D(T), \\ & \|L\{|U_0|^{p-1} |\partial_i U_l - \partial_i U_{l-1}|\}\| \\ & \leq C_{n,1,p} k^2 \|\tau_+^{(n-1)/2} w^{-1} U_0\|^{p-1} \|\partial_i U_l - \partial_i U_{l-1}\| E_1(T), \\ & \|L\{|U_l - U_{l-1}|^{p-1} |\partial_i U_{l-1}|\}\| \leq Ck^2 \|U_l - U_{l-1}\|^{p-1} \|\partial_i U_{l-1}\| D(T), \\ & \|L\{|U_l - U_{l-1}|^{p-1} |\partial_i U_0|\}\| \\ & \leq C_{n,p-1,p} k^2 \|U_l - U_{l-1}\|^{p-1} \|\tau_+^{(n-1)/2} w^{-1} \partial_i U_0\| E_{p-1}(T). \end{aligned}$$

Therefore, due to (5.13), all the assumptions imply that

$$\begin{aligned} & \|\partial_i U_{l+1} - \partial_i U_l\| \\ & \leq 2^{p-1} pAk^2 \{C\|W_l\|^{p-1} D(T) + C_{n,1,p} \|\tau_+^{(n-1)/2} w^{-1} W_0\|^{p-1} E_1(T)\} \times \\ & \quad \times \|\partial_i U_l - \partial_i U_{l-1}\| \\ & \quad + pAk^2 \{C\|\partial W_{l-1}\| D(T) + C_{n,p-1,p} \|\tau_+^{(n-1)/2} w^{-1} \partial W_0\| E_{p-1}(T)\} \times \\ & \quad \times \|U_l - U_{l-1}\|^{p-1} \\ & \leq 2^{p-1} pAk^2 \{C(2M_0 \varepsilon^p)^{p-1} D(T) + C_{n,1,p} (C_0 \varepsilon)^{p-1} E_1(T)\} \|\partial_i U_l - \partial_i U_{l-1}\| \\ & \quad + pAk^2 \{C(2M_0 \varepsilon^p) D(T) + C_{n,p-1,p} C_0 \varepsilon E_{p-1}(T)\} (\|U_2 - U_1\| 2^{-(l-1)})^{p-1}. \end{aligned}$$

This inequality yields

$$\|\partial_i U_{l+1} - \partial_i U_l\| \leq \frac{1}{2} \|\partial_i U_l - \partial_i U_{l-1}\| + \frac{N_0(\varepsilon, T)}{2^{(l-1)(p-1)}}, \quad (5.16)$$

where we set

$$N_0(\varepsilon, T) = pAk^2 \|U_2 - U_1\|^{p-1} \{C(2M_0 \varepsilon^p) D(T) + C_{n,p-1,p} C_0 \varepsilon E_{p-1}(T)\}$$

provided

$$2^{p-1}pAk^2\{C(2M_0)^{p-1}\varepsilon^{p(p-1)}D(T) + C_{n,1,p}(C_0)^{p-1}\varepsilon^{p-1}E_1(T)\} \leq \frac{1}{2}. \quad (5.17)$$

We note that (5.16) implies

$$\|\partial_i U_{l+1} - \partial_i U_l\| \leq \frac{1}{2^{l-1}}\|\partial_i U_2 - \partial_i U_1\| + \frac{N_0(\varepsilon, T)}{2^{(l-1)(p-1)}} \sum_{\nu=0}^{l-2} \frac{1}{2^\nu} \quad \text{for } l \geq 2.$$

The convergence of  $\{\partial_i U_l\}_{l \in \mathbf{N}}$  ( $i = 1, 2, 3, \dots, n$ ) follows from this estimate.

In this way, the convergence of  $\{U_l\}_{l \in \mathbf{N}}$  in  $Y_0 \subset X$  can be established if all the five conditions, (5.3), (5.7), (5.12), (5.15), (5.17), are satisfied. In order to complete the proof of Proposition 5.1, we shall fix  $\varepsilon_0 = \varepsilon_0(f, g, n, p, k)$  and  $c$  in the statement of Proposition 5.1. First, we propose a sufficient condition to (5.3), (5.7) as well as related factors in (5.12), (5.15), (5.17) to  $D(T)$  by

$$2^{2p}3^{p-1}pACk^2M_0^{p-1}\varepsilon^{p(p-1)}D(T) \leq 1. \quad (5.18)$$

Next, we propose sufficient conditions to related factors in (5.12), (5.15), (5.17) to  $E_1(T)$  and  $E_{p-1}(T)$  according to  $p$  by the following.

In the case of  $1 < p < \frac{n+1}{n-1}$ , such conditions are

$$2^{p+1}pAk^2C_{n,1,p}C_0^{p-1}\varepsilon^{p-1} \left( \frac{2T+3k}{k} \right)^{-q} \leq 1 \quad (5.19)$$

and

$$1 \geq 2^{2p-1}pAk^2C_{n,p-1,p}M_0^{p-2}C_0\varepsilon^{(p-1)^2} \left( \frac{2T+3k}{k} \right)^{-(p-1)q}. \quad (5.20)$$

If we put

$$c = \min \left\{ (2^{2p}3^{p-1}pACk^2(M_0)^{p-1})^{-1}, (2^{p+1}pAk^2C_{n,1,p}C_0^{p-1})^{-p}, (2^{2p-1}pAk^2C_{n,p-1,p}M_0^{p-2}C_0)^{-p/(p-1)} \right\} > 0,$$

the inequality  $\varepsilon^{p(p-1)}D(T) \leq c$  implies (5.18), (5.19) and (5.20), because of  $-pq < \gamma(n, p)/2$ . Furthermore, one can readily check that  $\varepsilon_0 = 1$  by (5.3).

In the case of  $p = \frac{n+1}{n-1}$ , let us fix  $\delta$  with

$$0 < \delta < \frac{1}{p}. \quad (5.21)$$



Then, similarly to the above, our conditions are

$$2^{p+1}pAC_{n,1,p}k^2C_0^{p-1}\varepsilon^{p-1}\left(\frac{2T+3k}{k}\right)^\delta \leq 1 \quad (5.22)$$

and

$$1 \geq 2^{2p-1}pAk^2C_{n,p-1,p}M_0^{p-2}C_0\varepsilon^{(p-1)^2}\left(\frac{2T+3k}{k}\right)^{\delta(p-1)}. \quad (5.23)$$

If we put

$$c = \min \left\{ (2^{2p}3^{p-1}pACk^2(M_0)^{p-1})^{-1}, (2^{p+1}pAk^2C_{n,1,p}C_0^{p-1})^{-p}, (2^{2p-1}pAk^2C_{n,p-1,p}M_0^{p-2}C_0)^{-p/(p-1)} \right\} > 0,$$

the inequality  $\varepsilon^{p(p-1)}D(T) \leq c$  implies (5.18), (5.22) and (5.23), because of (5.21). Furthermore, one can readily check that  $\varepsilon_0 = 1$  by (5.3).

Finally, in the case of  $p > \frac{n+1}{n-1}$ , our conditions are

$$2^{p+1}pAk^2C_{n,1,p}C_0^{p-1}\varepsilon^{p-1} \leq 1 \quad (5.24)$$

and

$$2^{2p-1}pAk^2C_{n,p-1,p}M_0^{p-2}C_0\varepsilon^{(p-1)^2} \leq 1. \quad (5.25)$$

If we put

$$c = (2^{2p}3^{p-1}pACk^2(M_0)^{p-1})^{-1},$$

the inequality  $\varepsilon^{p(p-1)}D(T) \leq c$  implies (5.18). Furthermore, one can find that

$$\varepsilon_0 = \min \left\{ 1, (2^{p+1}pAk^2C_{n,1,p})^{-1/(p-1)}C_0^{-1}, (2^{2p-1}pAk^2M_0^{p-2}C_{n,p-1,p}C_0)^{-1/(p-1)^2} \right\} > 0,$$

by (5.3), (5.24) and (5.25). Therefore, the proof of proposition 5.1 is completed.  $\square$

## 6 Lower bound in even space dimensions

Similarly to the previous section, we investigate the lower bound of the lifespan in the even dimensional case. Our purpose is to show the following proposition.

**Proposition 6.1** *Let  $n = 4, 6, 8, \dots$ . Suppose that the same assumptions in Theorem 2.1, (2.19) and (2.20), are fulfilled. Then, there exists a positive constant  $\varepsilon_0 = \varepsilon_0(f, g, n, p, k)$  such that each of (2.14) and (2.18) admits a unique classical solution  $u \in C^2(\mathbf{R}^4 \times [0, T])$  if  $n = 4$  and  $p = p_0(4) = 2$ , or each of (2.12) and (2.17) admits a unique solution  $u \in C^1(\mathbf{R}^n \times [0, T])$  otherwise, as far as  $T$  satisfies*

$$T \leq \begin{cases} c\varepsilon^{-2p(p-1)/\gamma(p,n)} & \text{if } 1 < p < p_0(n) \\ \exp(c\varepsilon^{-p(p-1)}) & \text{if } p = p_0(n) \end{cases} \quad (6.1)$$

for  $0 < \varepsilon \leq \varepsilon_0$  and  $c$  is a positive constant independent of  $\varepsilon$ .

Employing the similar argument to odd dimensions, we shall construct a  $C^1$  solution of the integral equation (3.8) as a limit of  $\{U_l\}_{l \in \mathbf{N}}$  in  $X$ . We also remark that it is possible to construct a  $C^2$  solution if and only if  $(n, p) = (4, 2)$  in our problem. However, its construction is almost the same as for  $C^1$  solution. Therefore we shall omit it. Now, define a closed subspace  $Y_1$  in  $X$  by

$$Y_1 = \{U \in X : \|\nabla_x^\alpha U\| \leq M_1 \varepsilon^p \text{ } (|\alpha| \leq 1)\},$$

where  $M_1$  is defined by

$$M_1 = 2^{2p} p A k^2 (C_{n,0,p} C_1^p + C_{n,1,p} C_1^{p-1} C_2 + C_{n,p-1,p} C_1 C_2^{p-1} + C_{n,0,p} C_2^p) > 0.$$

Recall that  $A$  is the one in (2.19) and that  $C_{n,0,p}$ ,  $C_{n,1,p}$  and  $C_{n,p-1,p}$  are the one in (4.5) and (4.9).  $C_1$  and  $C_2$  are positive constants independent of  $\varepsilon$  which satisfy that

$$\left\| (\tau_+ \tau_-)^{(n-1)/2} \frac{\partial W_{00}}{w} \right\| \leq C_1 \varepsilon, \quad \left\| (\tau_+ \tau_-)^{(n-1)/2} \frac{\partial W_{01}}{w \tau_-} \right\| \leq C_2 \varepsilon^p. \quad (6.2)$$

The existence of  $C_1$  and  $C_2$  is trivial by the definition of the weight function in (3.11) and the decay estimate for  $U_{00} = v_0$  and  $U_{01} = v_1$  in (3.6) and (3.7).

**Proof of the Proposition 6.1.** First of all, we shall show the convergence of  $\{U_l\}_{l \in \mathbf{N}}$ . The boundedness of  $\{U_l\}_{l \in \mathbf{N}}$  in  $Y_1$ ,

$$\|U_l\| \leq 2M_1 \varepsilon^p \text{ } (l \in \mathbf{N}), \quad (6.3)$$

can be obtained by induction with respect to  $l$  as follows. Similarly to (5.5), it follows from (3.9) and (4.5) of Lemma 4.3 with  $\nu = 0$  and  $a = 0, 1$  that

$$\begin{aligned} \|U_1\| \leq & 2^{2p} A k^2 C_{n,0,p} \left( \left\| (\tau_+ \tau_-)^{(n-1)/2} \frac{U_{00}}{w} \right\|^p E_{0,0}(T) + \right. \\ & \left. + \left\| (\tau_+ \tau_-)^{(n-1)/2} \frac{U_{01}}{w \tau_-} \right\|^p E_{0,1}(T) \right). \end{aligned} \quad (6.4)$$

By the definition of (4.6), (4.7) and (4.8) with  $\nu = a = 0$ , we get

$$E_{0,0}(T) = 1. \quad (6.5)$$

It follows from (6.2) that

$$\|U_1\| \leq 2^{2p} A k^2 C_{n,0,p} \varepsilon^p (C_1^p + C_2^p \varepsilon^{p(p-1)} E_{0,1}(T)).$$

This inequality show  $\|U_1\| \leq M_1 \varepsilon^p$  provided

$$\varepsilon^{p(p-1)} E_{0,1}(T) \leq 1. \quad (6.6)$$

Assume that  $\|U_{l-1}\| \leq 2M_1 \varepsilon^p$  ( $l \geq 2$ ). Making use of (5.6), the assumption of the induction and (4.1) yield that

$$\begin{aligned} \|U_l\| &\leq 2^p A k^2 \left\{ C \|U_{l-1}\|^p D(T) + C_{n,0,p} \left\| (\tau_+ \tau_-)^{(n-1)/2} \frac{U_{00}}{w} \right\|^p E_{0,0}(T) \right. \\ &\quad \left. + C_{n,0,p} \left\| (\tau_+ \tau_-)^{(n-1)/2} \frac{U_{01}}{w \tau_-} \right\|^p E_{0,1}(T) \right\} \\ &\leq 2^p A C k^2 (2M_1 \varepsilon^p)^p D(T) + M_1 \varepsilon^p. \end{aligned}$$

This inequality shows (6.3) provided

$$2^p A C k^2 (2M_1)^p \varepsilon^{p^2} D(T) \leq M_1 \varepsilon^p. \quad (6.7)$$

Next we shall estimate the differences under (6.6) and (6.7) ensuring the boundedness (6.3). We note that (6.3) implies  $\|W_l\| \leq 2M_1 \varepsilon^p$  ( $l \in \mathbf{N}$ ). Making use of the inequalities (5.8) and (5.9), we have

$$|U_{l+1} - U_l| \leq 2^{p-1} p A L \{ \{|3W_l|^{p-1} + 2^{p-1}(|U_{00}|^{p-1} + |U_{01}|^{p-1})\} |U_l - U_{l-1}| \}.$$

Applying (4.5) of Lemma 4.3 with  $\nu = 1$  and  $a = 0, 1$ , we get

$$\begin{aligned} &\|L(|U_{0a}|^{p-1} |U_l - U_{l-1}|)\| \\ &\leq k^2 C_{n,1,p} E_{1,a}(T) \left\| (\tau_- \tau_+)^{(n-1)/2} \frac{U_{0a}}{w \tau_-^a} \right\|^{p-1} \|U_l - U_{l-1}\| \end{aligned} \quad (6.8)$$

for  $a = 0, 1$ . Since (6.3) implies that  $\|W_l\| \leq 2M_1 \varepsilon^p$  for  $l \geq 2$ , the convergence of  $\{U_l\}_{l \in \mathbf{N}}$  follows from

$$\|U_{l+1} - U_l\| \leq \frac{1}{2} \|U_l - U_{l-1}\| \quad \text{for } l \geq 2$$

provided

$$\begin{aligned} &2^{p-1} p A k^2 \{ C (6M_1 \varepsilon^p)^{p-1} D(T) + 2^{p-1} C_{n,1,p} (C_1 \varepsilon)^{p-1} E_{1,0}(T) \\ &\quad + 2^{p-1} C_{n,1,p} C_2^{p-1} E_{1,1}(T) \varepsilon^{p(p-1)} \} \leq \frac{1}{2}, \end{aligned} \quad (6.9)$$

Thus, we obtain

$$\|U_{l+1} - U_l\| \leq \frac{1}{2^{l-1}} \|U_2 - U_1\| \quad \text{for } l \geq 2 \quad (6.10)$$

which implies the convergence of  $\{U_l\}_{l \in \mathbf{N}}$ .

Now we shall show the convergence of  $\{\partial_i U_l\}_{l \in \mathbf{N}}$  for  $i = 1, 2, \dots, n$  under the conditions, (6.6), (6.7), (6.9) which ensure the convergence of  $\{U_l\}_{l \in \mathbf{N}}$ . As before, the boundedness

$$\|\partial_i U_l\| \leq 2M_1 \varepsilon^p \quad (l \in \mathbf{N}, i = 1, 2, \dots, n) \quad (6.11)$$

can be obtained by induction as follows. Similarly to (6.4), we have

$$\begin{aligned} \|\partial_i U_1\| &\leq 2^p p A \|L(|U_0|^{p-1} |\partial_i U_0|)\| \\ &\leq 2^{2p-1} p A \|L((|U_{00}|^{p-1} + |U_{01}|^{p-1})(|\partial_i U_{00}| + |\partial_i U_{01}|))\| \\ &\leq 2^{2p-1} p A \|L(\partial W_{00}^p + W_{00}^{p-1} \partial W_{01} + W_{01}^{p-1} \partial W_{00} + \partial W_{01}^p)\|. \end{aligned}$$

Applying (4.9) of Lemma 4.3 with  $\kappa = 1$  and  $\kappa = p - 1$ , we get

$$\begin{aligned} &\|L(W_{00}^{p-1} \partial W_{01})\| \\ &\leq C_{n,1,p} k^2 \left\| (\tau_- \tau_+)^{(n-1)/2} \frac{W_{00}}{w} \right\|^{p-1} \left\| (\tau_- \tau_+)^{(n-1)/2} \frac{\partial W_{01}}{w \tau_-} \right\| F_1(T) \end{aligned}$$

and

$$\begin{aligned} &\|L(W_{01}^{p-1} \partial W_{00})\| \\ &\leq C_{n,p-1,p} k^2 \left\| (\tau_- \tau_+)^{(n-1)/2} \frac{W_{01}}{w \tau_-} \right\|^{p-1} \left\| (\tau_- \tau_+)^{(n-1)/2} \frac{\partial W_{00}}{w} \right\| F_{p-1}(T). \end{aligned}$$

By virtue of (6.2) and (6.5), we obtain

$$\begin{aligned} \|\partial_i U_1\| &\leq 2^{2p-1} p A k^2 \varepsilon^p \left\{ C_{n,0,p} C_1^p + C_{n,1,p} C_1^{p-1} C_2 \varepsilon^{p-1} F_1(T) \right. \\ &\quad \left. + C_{n,p-1,p} C_2^{p-1} C_1 \varepsilon^{(p-1)^2} F_{p-1}(T) \right. \\ &\quad \left. + C_{n,0,p} C_2^p \varepsilon^{p(p-1)} E_{0,1}(T) \right\}. \end{aligned}$$

This inequality shows  $\|\partial_i U_1\| \leq M_1 \varepsilon^p$  provided (6.6),

$$\varepsilon^{p-1} F_1(T) \leq 1 \quad (6.12)$$

and

$$\varepsilon^{(p-1)^2} F_{p-1}(T) \leq 1 \quad (6.13)$$

hold.

Assume that  $\|\partial_i U_{l-1}\| \leq 2M_1 \varepsilon^p$  ( $l \geq 2$ ). Then we get

$$\begin{aligned} |\partial_i U_l| &\leq pA|L(|U_{l-1} + U_0|^{p-1}|\partial_i(U_{l-1} + U_0)|)| \\ &\leq 2^{p-1}pAL\{(|U_{l-1}|^{p-1} + |U_0|^{p-1})(|\partial_i U_{l-1}| + |\partial_i U_0|)\} \\ &\leq 2^{p-1}pAL\{(|U_{l-1}|^{p-1} + 2^{p-1}(|U_{00}|^{p-1} + |U_{01}|^{p-1})) \\ &\quad \times (|\partial_i U_{l-1}| + |\partial_i U_{00}| + |\partial_i U_{01}|)\}. \end{aligned}$$

Similarly to (6.8), we obtain that

$$\begin{aligned} &\|L(|U_{l-1}|^{p-1}|\partial_i U_{0a}|)\| \\ &\leq C_{n,p-1,p}k^2\|U_{l-1}\|^{p-1}\left\|(\tau_-\tau_+)^{(n-1)/2}\frac{\partial_i U_{0a}}{w\tau_-^a}\right\|E_{p-1,a}(T), \\ &\|L(|U_{0a}|^{p-1}|\partial_i U_{l-1}|)\| \\ &\leq C_{n,1,p}k^2\|\partial_i U_{l-1}\|\left\|(\tau_-\tau_+)^{(n-1)/2}\frac{U_{0a}}{w\tau_-^a}\right\|^{p-1}E_{1,a}(T) \end{aligned}$$

for  $a = 0, 1$ . Making use of (6.2) and (6.5), we get

$$\begin{aligned} &\|\partial_i U_l\| \\ &\leq 2^{p-1}pAk^2[C(2M_1\varepsilon^p)^pD(T) + C_{n,p-1,p}(2M_1\varepsilon^p)^{p-1}C_1\varepsilon E_{p-1,0}(T) \\ &\quad + C_{n,p-1,p}(2M_1\varepsilon^p)^{p-1}C_2\varepsilon^p E_{p-1,1}(T) \\ &\quad + 2^{p-1}\{C_{n,1,p}(C_1\varepsilon)^{p-1}(2M_1\varepsilon^p)E_{1,0}(T) + C_{n,0,p}C_1^p\varepsilon^p \\ &\quad + C_{n,1,p}(C_1\varepsilon)^{p-1}C_2\varepsilon^p F_1(T) + C_{n,1,p}(C_2\varepsilon^p)^{p-1}(2M_1\varepsilon^p)E_{1,1}(T) \\ &\quad + C_{n,p-1,p}(C_2\varepsilon^p)^{p-1}C_1\varepsilon F_{p-1}(T) + C_{n,0,p}(C_2\varepsilon^p)^p E_{0,1}(T)\}]. \end{aligned}$$

Under the assumptions (6.6), (6.12) and (6.13), this inequality shows (6.11) provided

$$\begin{aligned} &(7/4)M_1\varepsilon^p \\ &\geq 2^{p-1}pAk^2[C(2M_1)^p\varepsilon^{p^2}D(T) \\ &\quad + C_{n,p-1,p}(2M_1)^{p-1}C_1\varepsilon^{p^2-p+1}E_{p-1,0}(T) \\ &\quad + C_{n,p-1,p}(2M_1)^{p-1}C_2\varepsilon^{p^2}E_{p-1,1}(T) + 2^{p-1}\{C_{n,1,p}C_1^{p-1}\varepsilon^{2p-1} \\ &\quad \times 2M_1E_{1,0}(T) + C_{n,1,p}C_2^{p-1}2M_1\varepsilon^{p^2}E_{1,1}(T)\}]. \end{aligned} \tag{6.14}$$

Next we shall estimate the differences under the conditions, (6.6), (6.7), (6.9), (6.12), (6.13), (6.14) which ensure the convergence of  $\{U_l\}_{l \in \mathbf{N}}$  and the boundedness of  $\{\partial U_l\}_{l \in \mathbf{N}}$ . Similarly to odd dimensions, we have

$$\begin{aligned} &|\partial_i U_{l+1} - \partial_i U_l| \\ &\leq 2^{p-1}pAL\{(|U_l|^{p-1} + 2^{p-1}\{|U_{00}|^{p-1} + |U_{01}|^{p-1}\})|\partial_i U_l - \partial_i U_{l-1}|\} \\ &\quad + pAL\{|U_l - U_{l-1}|^{p-1}(|\partial_i U_{l-1}| + |\partial_i U_{00}| + |\partial_i U_{01}|)\}. \end{aligned}$$

Applying (4.5) of Lemma 4.3 with  $\nu = 1$ ,  $\nu = p - 1$  and  $a = 0, 1$ , we get

$$\begin{aligned}
& \|L\{|U_{0a}|^{p-1}|\partial_i U_l - \partial_i U_{l-1}|\}\| \\
& \leq C_{n,1,p} k^2 \left\| (\tau_- \tau_+)^{(n-1)/2} \frac{U_{0a}}{w \tau_-^a} \right\|^{p-1} \|\partial_i U_l - \partial_i U_{l-1}\| E_{1,a}(T), \\
& \|L\{|U_l - U_{l-1}|^{p-1}|\partial_i U_{0a}|\}\| \\
& \leq C_{n,p-1,p} k^2 \|U_l - U_{l-1}\|^{p-1} \left\| (\tau_- \tau_+)^{(n-1)/2} \frac{\partial_i U_{0a}}{w \tau_-^a} \right\| E_{p-1,a}(T)
\end{aligned}$$

for  $a = 0, 1$ . Then, all the assumptions imply that

$$\begin{aligned}
& \|\partial_i U_{l+1} - \partial_i U_l\| \\
& \leq 2^{p-1} p A k^2 [\{C(2M_1 \varepsilon^p)^{p-1} D(T) + 2^{p-1} \{C_{n,1,p}(C_1 \varepsilon)^{p-1} E_{1,0}(T) \\
& \quad + C_{n,1,p}(C_2 \varepsilon^p)^{p-1} E_{1,1}(T)\}\} \|\partial_i U_l - \partial_i U_{l-1}\| \\
& \quad + p A k^2 \{C(2M_1 \varepsilon^p) D(T) + C_{n,p-1,p} C_1 \varepsilon E_{p-1,0}(T) \\
& \quad + C_{n,p-1,p} C_2 \varepsilon^p E_{p-1,1}(T)\} (\|U_2 - U_1\| 2^{-(l-1)})^{p-1}.
\end{aligned}$$

This inequality yields

$$\|\partial_i U_{l+1} - \partial_i U_l\| \leq \frac{1}{2} \|\partial_i U_l - \partial_i U_{l-1}\| + \frac{N_1(\varepsilon, T)}{2^{(l-1)(p-1)}}, \quad (6.15)$$

where we set

$$\begin{aligned}
N_1(\varepsilon, T) &= p A k^2 \{C(2M_1 \varepsilon^p) D(T) + C_{n,p-1,p} C_1 \varepsilon E_{p-1,0}(T) \\
&\quad + C_{n,p-1,p} C_2 \varepsilon^p E_{p-1,1}(T)\}
\end{aligned}$$

provided

$$\begin{aligned}
& 2^{p-1} p A k^2 [\{C(2M_1)^{p-1} \varepsilon^{p(p-1)} D(T) \\
& \quad + 2^{p-1} \{C_{n,1,p}(C_1)^{p-1} \varepsilon^{p-1} E_{1,0}(T) \\
& \quad + C_{n,1,p} C_2^{p-1} \varepsilon^{p(p-1)} E_{1,1}(T)\}\}] \leq \frac{1}{2}.
\end{aligned} \quad (6.16)$$

Hence, the convergence of  $\{\partial_i U_l\}_{l \in \mathbf{N}}$  ( $i = 1, 2, 3, \dots, n$ ) follows from this estimate.

In this way, the convergence of  $\{U_l\}_{l \in \mathbf{N}}$  in  $Y_0 \subset X$  can be established if all the seven conditions, (6.6), (6.7), (6.9), (6.12), (6.13), (6.14), (6.16) are satisfied. In order to complete the proof of Proposition 6.1, we shall fix  $\varepsilon_0 = \varepsilon_0(f, g, n, p, k)$  and  $c$  in the statement of Proposition 6.1. First, we propose a sufficient condition to (6.7) as well as related factors in (6.9), (6.14), (6.16) to  $D(T)$  as

$$2^{2p-1} 3^p p A C k^2 M_1^{p-1} \varepsilon^{p(p-1)} D(T) \leq 1. \quad (6.17)$$

Next we propose sufficient conditions to related factors in (6.6), (6.9), (6.12), (6.13), (6.14) and (6.16) to

$$E_{0,1}(T), E_{1,0}(T), E_{1,1}(T), F_1(T), F_{p-1}(T), E_{p-1,0}(T), E_{p-1,1}(T)$$

up to  $p$ .

**Conditions in the case of  $1 < p < \frac{n+1}{n-1}$ .**

(i) Conditions from  $E_{0,1}(T)$ .

In (4.8), setting  $\nu = 0$  and  $a = 1$ , we have that  $\sigma = -(n-3)p/2 < -1$  when  $n \geq 6$  and  $\sigma = \mu = -p/2 > -1$  when  $n = 4$ , which imply

$$E_{0,1}(T) = \begin{cases} 1 & \text{if } n \geq 6, \\ \left(\frac{2T+3k}{k}\right)^{1-p/2} & \text{if } n = 4. \end{cases} \quad (6.18)$$

Since  $E_{0,1}(T)$  appears in (6.6), the conditions are

$$\varepsilon^{p(p-1)} \leq 1 \quad \text{if } n \geq 6, \quad (6.19)$$

$$\left(\frac{2T+3k}{k}\right)^{1-p/2} \varepsilon^{p(p-1)} \leq 1 \quad \text{if } n = 4. \quad (6.20)$$

(ii) Condition from  $E_{1,0}(T)$ .

In (4.8), setting  $\nu = 1$  and  $a = 0$ , we have that  $\sigma = -(n-1)(p-1)/2 > -1$  and  $\mu = -2q - 1$ , which imply

$$E_{1,0}(T) = \left(\frac{2T+3k}{k}\right)^{-2q}.$$

Since  $E_{1,0}(T)$  appears in (6.9), (6.14) and (6.16), the condition is

$$2^{2p-1} 3pAk^2 C_{n,1,p} C_1^{p-1} \varepsilon^{p-1} \left(\frac{2T+3k}{k}\right)^{-2q} \leq 1. \quad (6.21)$$

(iii) Condition from  $E_{1,1}(T)$ .

In (4.8), setting  $\nu = a = 1$ , we have that  $\sigma = -(n-3)(p-1)/2 > -1$  and  $\mu = n-1-(n-2)p$ , which imply

$$E_{1,1}(T) = \left(\frac{2T+3k}{k}\right)^{n-(n-2)p}.$$

Since  $E_{1,1}(T)$  appears in (6.9), (6.14) and (6.16), the condition is

$$2^{2p-1} 3pAk^2 C_{n,1,p} C_2^{p-1} \varepsilon^{p(p-1)} \left(\frac{2T+3k}{k}\right)^{n-(n-2)p} \leq 1. \quad (6.22)$$

(iv) Conditions from  $F_1(T)$ .

In (4.10), setting  $\nu = 1$ , we have that

$$\begin{aligned} \kappa &= 1 - \frac{n-1}{2}p < -1 && \text{if } n \geq 6, \text{ or } n = 4 \text{ and } p > \frac{4}{3}, \\ \kappa &= -1 && \text{if } n = 4 \text{ and } p = \frac{4}{3}, \\ \kappa &> -1 && \text{if } n = 4 \text{ and } 1 < p < \frac{4}{3}, \end{aligned}$$

which imply

$$F_1(T) = \begin{cases} 1 & \text{if } n \geq 6, \text{ or } n = 4 \text{ and } p > \frac{4}{3}, \\ \log \frac{2T+3k}{k} & \text{if } n = 4 \text{ and } p = \frac{4}{3}, \\ \left( \frac{2T+3k}{k} \right)^{2-3p/2} & \text{if } n = 4 \text{ and } 1 < p < \frac{4}{3}. \end{cases} \quad (6.23)$$

Since  $F_1(T)$  appears in (6.12), the conditions are

$$\varepsilon^{p(p-1)} \leq 1 \quad \text{if } n \geq 6, \text{ or } n = 4 \text{ and } p > \frac{4}{3}, \quad (6.24)$$

$$\log \frac{2T+3k}{k} \varepsilon^{p(p-1)} \leq 1 \quad \text{if } n = 4 \text{ and } p = \frac{4}{3}, \quad (6.25)$$

$$\left( \frac{2T+3k}{k} \right)^{2-3p/2} \varepsilon^{p(p-1)} \leq 1 \quad \text{if } n = 4 \text{ and } 1 < p < \frac{4}{3}. \quad (6.26)$$

(v) Condition from  $F_{p-1}(T)$ .

In (4.10), setting  $\nu = p-1$ , we have  $\kappa = -(n-3)p/2 - 1 < -1$ , which implies  $F_{p-1}(T) = 1$ . Since  $F_{p-1}(T)$  appears in (6.13), the condition is

$$\varepsilon^{p(p-1)} \leq 1. \quad (6.27)$$

(vi) Condition from  $E_{p-1,0}(T)$ .

In (4.8), setting  $\nu = p-1$  and  $a = 0$ , we have  $\sigma = -(n-1)/2 < -1$ , which implies

$$E_{p-1,0}(T) = \left( \frac{2T+3k}{k} \right)^{-(p-1)q}.$$

Since  $E_{p-1,0}(T)$  appears in (6.14), the condition is

$$2^{2p5} \cdot 7^{-1} p A k^2 C_{n,p-1,p} C_1 M_1^{p-2} \varepsilon^{(p-1)^2} \left( \frac{2T+3k}{k} \right)^{-(p-1)q} \leq 1. \quad (6.28)$$



(vii) Conditions from  $E_{p-1,1}(T)$ .

In (4.8), setting  $\nu = p - 1$  and  $a = 1$ , we have that  $\sigma = -(n - 3)/2 < -1$  when  $n \geq 6$  and  $\sigma > -1, \mu = -1/2 - (p - 1)q$  when  $n = 4$ , which imply

$$E_{p-1,1}(T) = \begin{cases} \left(\frac{2T+3k}{k}\right)^{-(p-1)q} & \text{if } n \geq 6, \\ \left(\frac{2T+3k}{k}\right)^{1/2-(p-1)q} & \text{if } n = 4. \end{cases} \quad (6.29)$$

Since  $E_{p-1,1}(T)$  appear in (6.14), the conditions are

$$\begin{aligned} & 2^{2p}5 \cdot 7^{-1}pAk^2C_{n,p-1,p}C_2M_1^{p-2}\varepsilon^{p(p-1)} \\ & \times \left(\frac{2T+3k}{k}\right)^{-(p-1)q} \leq 1 \quad \text{if } n \geq 6, \end{aligned} \quad (6.30)$$

$$\begin{aligned} & 2^{2p}5 \cdot 7^{-1}pAk^2C_{4,p-1,p}C_2M_1^{p-2}\varepsilon^{p(p-1)} \\ & \times \left(\frac{2T+3k}{k}\right)^{1/2-(p-1)q} \leq 1 \quad \text{if } n = 4. \end{aligned} \quad (6.31)$$

Now, we are in a position to summarize all the conditions in (i)-(vii) above. Set

$$\varepsilon_0 = 1 \quad (6.32)$$

Then, (6.32) implies (6.19), (6.24) and (6.27). In order to make that (6.17) includes (6.25), we employ Lemma 4.9 with  $\delta = \gamma(p, 4)/2 > 0$  and  $X = (2T + 3k)/k > 1$ . Then, if we set

$$\begin{aligned} c = \min \{ & (2^{2p-1}3^p p A C k^2 (M_1)^{p-1})^{-1}, 1, (2^{2p-1}3^p A k^2 C_{n,1,p} C_1^{p-1})^{-p}, \\ & (2^{2p-1}3^p A k^2 C_{n,1,p} C_2^{p-1})^{-1}, \gamma(4/3, 4)/2, \\ & (2^{2p}5 \cdot 7^{-1}p A k^2 C_{n,p-1,p} C_1 M_1^{p-2})^{-p/(p-1)}, \\ & (2^{2p}5 \cdot 7^{-1}p A k^2 C_{n,p-1,p} C_2 M_1^{p-2})^{-1} \} > 0, \end{aligned}$$

the inequality  $\varepsilon^{p(p-1)}D(T) \leq c$  implies (6.17), (6.20), (6.21), (6.22), (6.25), (6.26), (6.28), (6.30) and (6.31) because of  $1 - p/2 \leq \gamma(p, 4)/2$  in (6.20),  $-2pq \leq \gamma(p, n)/2$  in (6.21),  $n - (n - 2)p \leq \gamma(p, n)/2$  in (6.22),  $2 - 3p/2 \leq \gamma(p, 4)/2$  in (6.26),  $-pq \leq \gamma(p, n)/2$  in (6.28),  $-(p - 1)q \leq \gamma(p, n)/2$  in (6.30) and  $1/2 - (p - 1)q \leq \gamma(p, 4)/2$  in (6.31).

**Conditions in the case of  $p = \frac{n+1}{n-1}$ .**

(i) Conditions from  $E_{0,1}(T)$ .

In (4.7), setting  $\nu = 0$  and  $a = 1$ , we have that  $\sigma = -(n-3)(n+1)/2(n-1) < -1$  when  $n \geq 6$  and  $\sigma > -1$  when  $n = 4$ , which imply

$$E_{0,1}(T) = \begin{cases} 1 & \text{if } n \geq 6, \\ \left(\frac{2T+3k}{k}\right)^{1-p/2} & \text{if } n = 4. \end{cases} \quad (6.33)$$

Since  $E_{0,1}(T)$  appears in (6.6), the conditions are

$$\varepsilon^{p(p-1)} \leq 1 \text{ if } n \geq 6, \quad (6.34)$$

$$\left(\frac{2T+3k}{k}\right)^{1-p/2} \varepsilon^{p(p-1)} \leq 1 \text{ if } n = 4. \quad (6.35)$$

(ii) Condition from  $E_{1,0}(T)$ .

In (4.7), setting  $\nu = 1$  and  $a = 0$ , we have  $\sigma = -(n-1)(p-1)/2 = -1$ , which implies

$$E_{1,0}(T) = \left(\frac{2T+3k}{k}\right)^\delta.$$

Since  $E_{1,0}(T)$  appears in (6.9), (6.14) and (6.16), the conditions is

$$2^{2p-1} 3p A k^2 C_{n,1,p} C_1^{p-1} \varepsilon^{p-1} \left(\frac{2T+3k}{k}\right)^\delta \leq 1. \quad (6.36)$$

(iii) Condition from  $E_{1,1}(T)$ .

In (4.7), setting  $\nu = a = 1$ , we have  $\sigma = -(n-3)/(n-1) > -1$ , which implies

$$E_{1,1}(T) = \left(\frac{2T+3k}{k}\right)^{2/(n-1)}.$$

Since  $E_{1,1}(T)$  appears in (6.9) (6.14) and (6.16), the condition is

$$2^{2p-1} 3p A k^2 C_{n,1,p} C_2^{p-1} \varepsilon^{p(p-1)} \left(\frac{2T+3k}{k}\right)^{2/(n-1)} \leq 1. \quad (6.37)$$

(iv) Condition from  $F_1(T)$ .

In (4.10), setting  $\nu = 1$ , we have  $\kappa < -1$ , which implies  $F_1(T) = 1$ . Since,  $F_1(T)$  appears in (6.12), the condition is

$$\varepsilon^{p(p-1)} \leq 1. \quad (6.38)$$

(v) Condition from  $F_{p-1,0}(T)$ .

In (4.10), setting  $\nu = p-1$ , we have  $\kappa < -1$ , which implies  $F_{p-1}(T) = 1$ . Since,  $F_{p-1}(T)$  appears in (6.13), the condition is (6.38).

(vi) Condition from  $E_{p-1,0}(T)$ .

In (4.7), setting  $\nu = p - 1$  and  $a = 0$ , we have  $\sigma = -(n - 1)/2 < -1$ , which implies

$$E_{p-1,0}(T) = \left( \frac{2T + 3k}{k} \right)^{(p-1)\delta}.$$

Since  $E_{p-1,0}(T)$  appears in (6.14), the condition is

$$2^{2p}5 \cdot 7^{-1}pAk^2C_{n,p-1,p}C_1M_1^{p-2}\varepsilon^{(p-1)^2} \left( \frac{2T + 3k}{k} \right)^{(p-1)\delta} \leq 1. \quad (6.39)$$

(vii) Conditions from  $E_{p-1,1}(T)$ .

In (4.7), setting  $\nu = p - 1$  and  $a = 1$ , we have that  $\sigma = -(n - 3)/2 < -1$  when  $n \geq 6$  and  $\sigma > -1$  when  $n = 4$ , which imply

$$E_{p-1,1}(T) = \begin{cases} \left( \frac{2T + 3k}{k} \right)^{(p-1)\delta} & \text{if } n \geq 6, \\ \left( \frac{2T + 3k}{k} \right)^{1/2} & \text{if } n = 4. \end{cases} \quad (6.40)$$

Since  $E_{p-1,1}(T)$  appears in (6.16), the conditions are

$$\begin{aligned} & 2^{2p}5 \cdot 7^{-1}pAk^2C_{n,p-1,p}C_2M_1^{p-2}\varepsilon^{p(p-1)} \\ & \times \left( \frac{2T + 3k}{k} \right)^{(p-1)\delta} \leq 1 \quad \text{if } n \geq 6, \end{aligned} \quad (6.41)$$

$$\begin{aligned} & 2^{2p}5 \cdot 7^{-1}pAk^2C_{4,p-1,p}C_2M_1^{p-2}\varepsilon^{p(p-1)} \\ & \times \left( \frac{2T + 3k}{k} \right)^{1/2} \leq 1. \quad \text{if } n = 4. \end{aligned} \quad (6.42)$$

Now, we are in a position to summarize all the conditions in (i)-(vii) above. First we note that (6.32) implies (6.34) and (6.38). Then, if we assume (5.21) and set

$$\begin{aligned} c = \min \{ & (2^{2p-1}3^p p A C k^2 (M_1)^{p-1})^{-1}, 1, (2^{2p-1}3^p p A k^2 C_{n,1,p} C_1^{p-1})^{-p}, \\ & (2^{2p-1}3^p A k^2 C_{n,1,p} C_2^{p-1})^{-1}, \\ & (2^{2p}5 \cdot 7^{-1} p A k^2 C_{n,p-1,p} C_1 M_1^{p-2})^{-p/(p-1)}, \\ & (2^{2p}5 \cdot 7^{-1} p A k^2 C_{n,p-1,p} C_2 M_1^{p-2})^{-1} \} > 0, \end{aligned}$$

the inequality  $\varepsilon^{p(p-1)}D(T) \leq c$  implies (6.17), (6.35), (6.36), (6.37), (6.39), (6.41) and (6.42) because of  $1 - p/2 \leq 1$  in (6.35),  $p\delta < 1$  in (6.36) and (6.39),  $2/(n - 1) < 1$  in (6.37) and  $(p - 1)\delta < 1$  in (6.41).

**Conditions in the case of  $p > \frac{n+1}{n-1}$ .**

(i) Conditions from  $E_{0,1}(T)$ .

In (4.6), setting  $\nu = 0$  and  $a = 1$ , we have that  $\mu = -(n-3)p/2 < -1$  when  $n \geq 6$  and  $\mu = -p/2 \geq -1$  when  $n = 4$ , which imply

$$E_{0,1}(T) = \begin{cases} 1 & \text{if } n \geq 6, \\ \left(\frac{2T+3k}{k}\right)^{1-p/2} & \text{if } n = 4, p < 2, \\ \log \frac{2T+3k}{k} & \text{if } n = 4, p = 2. \end{cases} \quad (6.43)$$

Since  $E_{0,1}(T)$  appear (6.6), the conditions are

$$\varepsilon^{p(p-1)} \leq 1 \quad \text{if } n \geq 6, \quad (6.44)$$

$$\left(\frac{2T+3k}{k}\right)^{1-p/2} \varepsilon^{p(p-1)} \leq 1 \quad \text{if } n = 4, p < 2, \quad (6.45)$$

$$\log \frac{2T+3k}{k} \varepsilon^{p(p-1)} \leq 1 \quad \text{if } n = 4, p = 2. \quad (6.46)$$

(ii) Condition from  $E_{1,0}(T)$ .

In (4.6), setting  $\nu = 1$  and  $a = 0$ , we have  $\mu = -(n-1)(p-1)/2 > -1$ , which implies  $E_{1,0}(T) = 1$ . Since  $E_{1,0}(T)$  appears in (6.9), (6.14) and (6.16), the condition is

$$2^{2p-1} 3p A k^2 C_{n,1,p} C_1^{p-1} \varepsilon^{p-1} \leq 1. \quad (6.47)$$

(iii) Conditions from  $E_{1,1}(T)$ .

In (4.6), setting  $\nu = a = 1$ , we have  $\mu = -(n-3)(p-1)/2 - q = n-1 - (n-2)p$ , which implies

$$E_{1,1}(T) = \begin{cases} 1 & \text{if } p > \frac{n}{n-2}, \\ \log \frac{2T+3k}{k} & \text{if } p = \frac{n}{n-2}, \\ \left(\frac{2T+3k}{k}\right)^{n-(n-2)p} & \text{if } p < \frac{n}{n-2}. \end{cases} \quad (6.48)$$

Since  $E_{1,1}(T)$  appears in (6.9), (6.14) and (6.16), the conditions are

$$\left. \begin{aligned} & 2^{2p-1} 3p A k^2 C_{n,1,p} C_2^{p-1} \varepsilon^{p(p-1)} \times \\ & \times \left(\frac{2T+3k}{k}\right)^{n-(n-2)p} \end{aligned} \right\} \leq 1 \quad \text{if } p < \frac{n}{n-2}, \quad (6.49)$$

$$2^{2p-1} 3p A k^2 C_{n,1,p} C_2^{p-1} \varepsilon^{p(p-1)} \log \frac{2T+3k}{k} \leq 1 \quad \text{if } p = \frac{n}{n-2}, \quad (6.50)$$

$$2^{2p-1} 3p A k^2 C_{n,1,p} C_2^{p-1} \varepsilon^{p(p-1)} \leq 1 \quad \text{if } p > \frac{n}{n-2}. \quad (6.51)$$

(iv) Condition from  $F_1(T)$  and  $F_{p-1}(T)$ .

In (4.10), setting  $\nu = 1$  and  $\nu = p - 1$ , we have  $\kappa < -1$ , which implies  $F_1(T) = F_{p-1}(T) = 1$ . Since  $F_1(T)$  or  $F_{p-1}(T)$  appears in (6.12) or (6.13) respectively, the condition is

$$\varepsilon^{p(p-1)} \leq 1. \quad (6.52)$$

(v) Condition from  $E_{p-1,0}(T)$ .

In (4.6), setting  $\nu = p - 1$  and  $a = 0$ , we have  $\mu = -(n - 1)/2 - q(p - 1) < -1$ , which implies  $E_{p-1,0}(T) = 1$ . Since  $E_{p-1,0}(T)$  appears in (6.14), the condition is

$$2^{2p}5 \cdot 7^{-1}pAk^2C_{n,p-1,p}C_1M_1^{p-2}\varepsilon^{(p-1)^2} \leq 1. \quad (6.53)$$

(vi) Conditions from  $E_{p-1,1}(T)$ .

In (4.6), setting  $\nu = p - 1$  and  $a = 1$ , we have that  $\mu = -(n - 3)/2 - (p - 1)q < -1$  when  $n \geq 6$  and  $\mu > -1$  when  $n = 4$ , which imply

$$E_{p-1,1}(T) = \begin{cases} 1 & \text{if } n \geq 6, \\ \left(\frac{2T+3k}{k}\right)^{1/2-(p-1)q} & \text{if } n = 4, p < 2, \\ \log \frac{2T+3k}{k} & \text{if } n = 4, p = 2. \end{cases} \quad (6.54)$$

Since  $E_{p-1,1}(T)$  appears in (6.16), the condition are

$$2^{2p}5 \cdot 7^{-1}pAk^2C_{n,p-1,p}C_2M_1^{p-2}\varepsilon^{p(p-1)} \leq 1 \quad \text{if } n \geq 6, \quad (6.55)$$

$$\left. \begin{aligned} &2^{2p}5 \cdot 7^{-1}pAk^2C_{4,p-1,p}C_2M_1^{p-2}\varepsilon^{p(p-1)} \times \\ &\quad \times \left(\frac{2T+3k}{k}\right)^{1/2-(p-1)q} \end{aligned} \right\} \leq 1 \quad \text{if } n = 4, p < 2, \quad (6.56)$$

$$\left. \begin{aligned} &2^{2p}5 \cdot 7^{-1}pAk^2C_{4,p-1,p}C_2M_1^{p-2}\varepsilon^{p(p-1)} \times \\ &\quad \times \log \frac{2T+3k}{k} \end{aligned} \right\} \leq 1 \quad \text{if } n = 4, p = 2. \quad (6.57)$$

Now, we are in a position to summarize all the conditions in (i)-(vi) above. Set

$$\varepsilon_0 = \min \begin{cases} 1, (2^{2p-1}3pAk^2C_{n,1,p}C_1^{p-1})^{-1/(p-1)}, \\ \{2^{2p-1}3pAk^2C_{n,1,p}C_2^{p-1}\}^{-1/p(p-1)}, \\ \{2^{2p}5 \cdot 7^{-1}pAk^2C_{n,p-1,p}C_1M_1^{p-2}\}^{-1/(p-1)^2}, \\ \{2^{2p}5 \cdot 7^{-1}pAk^2C_{n,p-1,p}C_2M_1^{p-2}\}^{-1/p(p-1)} \end{cases} > 0. \quad (6.58)$$

Then, (6.58) implies (6.44), (6.47), (6.51), (6.52), (6.53) and (6.55). In order to make that (6.50) includes (6.17) when  $n \geq 6$ , we employ the lemma 4.9

with  $\delta = \gamma(p, n)/2 > 0$  and  $X = (2T + 3k)/k > 1$ . Then, if we set

$$c = \min \left\{ (2^{2p-1} 3^p p A C k^2 (M_1)^{p-1})^{-1}, 1, (2^{2p-1} 3^p A k^2 C_{n,1,p} C_2^{p-1})^{-1}, \right. \\ \left. (2^{2p-1} 3^p A k^2 C_{n,1,p} C_2^{p-1})^{-1} \gamma(n/(n-2), n)/2, \right. \\ \left. (2^{2p} 5 \cdot 7^{-1} p A k^2 C_{4,p-1,p} C_2 M_1^{p-2})^{-1} \right\} > 0,$$

the inequality  $\varepsilon^{p(p-1)} D(T) \leq c$  implies (6.17), (6.45), (6.46), (6.49), (6.50) and (6.56) and (6.57), because of  $1 - p/2 \leq \gamma(p, 4)/2$  in (6.45),  $n - (n-2)p \leq \gamma(p, n)/2$  in (6.49), and  $1/2 - (p-1)q \leq \gamma(p, 4)/2$  in (6.56). Therefore the proof of proposition 6.1 is now completed.  $\square$

## 7 Upper bound of the lifespan for the critical case in odd dimensions

In this section, we prove Theorem 2.2 for the critical case in odd space dimensions. The proof is divided into three steps. In the first step, we get a point-wise estimate of the linear term  $u^0$  from below by means of the representation formula due to Rammaha [27, 28]. In the second step, we employ the comparison argument between the solution of integral equations (2.17) and the blowing-up solution of ODE basically introduced by Zhou [36], in order to overcome the difficulty in the critical case. In the last step, we also employ the slicing method introduced by Agemi, Kurokawa and Takamura [3] which helps us to show the blow-up of the solution of such a modified ODE arising from the high dimensional case.

**Proposition 7.1** *Suppose that the assumptions of Theorem 2.2 are fulfilled. Let  $u$  be a  $C^0$ -solution of (2.17) in  $\mathbf{R}^n \times [0, T]$ . Then, there exists a positive constant  $\varepsilon_0 = \varepsilon_0(g, n, p, k)$  such that  $T$  cannot be taken as*

$$T > \exp(c\varepsilon^{-p(p-1)}) \quad \text{if } p = p_0(n) \quad (7.1)$$

for  $0 < \varepsilon \leq \varepsilon_0$ , where  $c$  is a positive constant independent of  $\varepsilon$ .

**Proof.** First we note that one may assume that the solution of (2.17) is radially symmetric without loss of the generality. To see this, we employ the spherical mean which is defined by

$$\tilde{v}(r, t) = \frac{1}{\omega_n} \int_{|\omega|=1} v(r\omega, t) dS_\omega \quad (r > 0),$$

for  $v \in C(\mathbf{R}^n \times [0, \infty))$ . If we take the spherical mean of (2.17), we get

$$\tilde{u} = \varepsilon \tilde{u}^0 + L(\widetilde{|u|^p}).$$

Thanks to the fundamental identity for iterated spherical means, we have

$$\widetilde{L(|u|^p)} = L_{odd}(\widetilde{|u|^p}),$$

where  $L_{odd}$  is the one in (4.17). See 78-81pp. of John [13] for details. Thus, it follows from Jensen's inequality  $\widetilde{|u|^p} \geq |\widetilde{u}|^p$  ( $p > 1$ ) and the positivity of  $L_{odd}$  that

$$\widetilde{u} \geq \varepsilon u^0 + L_{odd}(|\widetilde{u}|^p).$$

We estimate  $\widetilde{u}$  from below all the time in this section, so that we may assume that the equality holds here.

Let  $u = u(r, t)$  be a  $C^0$ -solution of

$$u = \varepsilon u^0 + L_{odd}(|u|^p) \quad \text{in} \quad (0, \infty) \times [0, T] \quad (7.2)$$

which is associated by (2.17). Note that  $u^0 = u^0(r, t)$  is a solution of

$$\begin{cases} u_{tt}^0 - \frac{n-1}{r} u_r^0 - u_{rr}^0 = 0 & \text{in } (0, \infty) \times [0, \infty) \\ u^0(r, 0) = 0, \quad u_t^0(r, 0) = g(r), \quad r \in (0, \infty). \end{cases} \quad (7.3)$$

**[The 1st step] Estimate of  $u^0$ .**

We have the following representation of  $u^0$ .

**Lemma 7.1 (Rammaha [27])** *Let  $n = 5, 7, 9, \dots$  and  $u^0$  be a solution of (7.3). Then,  $u^0$  is represented by*

$$u^0(r, t) = \frac{1}{2r^{(n-1)/2}} \int_{|r-t|}^{r+t} \lambda^{(n-1)/2} g(\lambda) P_{(n-3)/2} \left( \frac{\lambda^2 + r^2 - t^2}{2r\lambda} \right) d\lambda,$$

where  $P_k$  is Legendre polynomial of degree  $k$  defined by

$$P_k(z) = \frac{1}{2^k k!} \frac{d^k}{dz^k} (z^2 - 1)^k.$$

See (6a) on 681p. in [27] for the proof. This lemma implies the following estimate.

**Lemma 7.2 (Rammaha [27])** *Let  $n = 5, 7, 9, \dots$ . Assume (2.22). Then there exists a positive constant  $C_g$  such that for  $t + k_0 < r < t + k_1$  and  $t \geq k_2$ ,*

$$u^0(r, t) \geq \frac{C_g}{r^{(n-1)/2}}, \quad (7.4)$$

where  $k_2 = k - k_0$ .

See Lemma 2 on 682p. in [27] for the proof.

In order to prove the blow up result, we employ the iteration argument originally introduced by John [14]. Our frame in the argument is obtained by the following lemma.

**Lemma 7.3** *Let  $u$  be a  $C^0$ -solution of (7.2). Assume (2.22). Then  $u$  in  $\Sigma_0 = \{(r, t) : 2k \leq t - r \leq r\}$  satisfies*

$$\begin{aligned} u(r, t) &\geq \frac{C2^{(n-1)/2}(t-r)^{(n-1)/2}}{r^{(3n-7)/2}} \times \\ &\times \iint_{R(r,t)} \{(t-r-\tau+\lambda)(t+r-\tau-\lambda)\}^{(n-3)/2} |u(\lambda, \tau)|^p d\lambda d\tau + \\ &+ \frac{E_1(t-r)^{(3n-5)/2-(n-1)p/2}}{r^{(3n-7)/2}} \varepsilon^p, \end{aligned} \quad (7.5)$$

where  $C$  is the one in (4.17),

$$E_1 = \frac{CC_g^p(k_1 - k_0)}{(n-1)2^{(n-1)p-(3n-9)/2}}$$

and

$$R(r, t) = \{(\lambda, \tau) : t-r \leq \lambda, \tau + \lambda \leq t+r, 2k \leq \tau - \lambda \leq t-r\}.$$

**Proof.** By virtue of Huygens' principle on  $u^0$  and (4.17), we have

$$u \geq I_1 + I_2 \quad \text{in } \Sigma_0,$$

where we set

$$\begin{aligned} I_1(r, t) &= Cr^{2-n} \int_{R(r,t)} (t-\tau)^{3-n} h(\lambda, t-\tau, r) \lambda |u(\lambda, \tau)|^p d\lambda d\tau, \\ I_2(r, t) &= Cr^{2-n} \int_{S(r,t)} (t-\tau)^{3-n} h(\lambda, t-\tau, r) \lambda |u(\lambda, \tau)|^p d\lambda d\tau, \\ S(r, t) &= \{(\lambda, \tau) : t-r \leq \lambda, \tau + \lambda \leq t+r, -k_1 \leq \tau - \lambda \leq -k_0\}. \end{aligned}$$

Changing variable by (4.28) in  $I_1$ , we have

$$\begin{aligned} I_1(r, t) &\geq \frac{Cr^{2-n}}{2} \int_{2k}^{t-r} \{(t-r-\beta)(t+r-\beta)\}^{(n-3)/2} d\beta \times \\ &\times \int_{2(t-r)+\beta}^{t+r} \{(\alpha - (t-r))(t+r-\alpha)\}^{(n-3)/2} \times \\ &\times (t - (\alpha + \beta)/2)^{3-n} (\alpha - \beta) |u(\lambda, \tau)|^p d\alpha \end{aligned}$$



in  $\Sigma_0$ . Noticing that

$$\begin{aligned} t + r - \beta &\geq 2r, \quad t - \frac{\alpha + \beta}{2} \leq r, \\ \alpha - \beta &\geq 2(t - r) \text{ and } \alpha - (t - r) \geq t - r \end{aligned}$$

hold in the domain of the integral above, we have

$$\begin{aligned} I_1(r, t) &\geq \frac{C2^{(n-3)/2}(t-r)^{(n-1)/2}}{r^{(3n-7)/2}} \int_{2k}^{t-r} (t-r-\beta)^{(n-3)/2} d\beta \times \\ &\times \int_{2(t-r)+\beta}^{t+r} (t+r-\alpha)^{(n-3)/2} |u(\lambda, \tau)|^p d\alpha \end{aligned}$$

in  $\Sigma_0$ . Hence, we have the first term of the right-hand side of (7.5).

Next, we shall show the second term of (7.5). Similarly to the above, we have

$$\begin{aligned} I_2(r, t) &\geq \frac{Cr^{2-n}}{2} \int_{-k_1}^{-k_0} \{(t-r-\beta)(t+r-\beta)\}^{(n-3)/2} d\beta \times \\ &\times \int_{2(t-r)+\beta}^{t+r} \{(\alpha - (t-r))(t+r-\alpha)\}^{(n-3)/2} \times \\ &\times (t - (\alpha + \beta)/2)^{3-n} (\alpha - \beta) |u(\lambda, \tau)|^p d\alpha \end{aligned}$$

in  $\Sigma_0$ . Note that

$$\begin{aligned} t + r - \beta &\geq r, \quad t - \frac{\alpha + \beta}{2} \leq r - \beta \leq 2r, \\ \alpha - (t - r) &\geq t - r + \beta \geq t - r - k \text{ and } t - r - \beta \geq t - r \end{aligned}$$

hold in the domain of the integral above. By making use of (7.4), we have

$$\begin{aligned} I_2(r, t) &\geq \frac{CC_g^p(t-r)^{(n-3)/2}(t-r-k)^{(n-3)/2}}{2^{n-2}r^{(3n-7)/2}} \varepsilon^p \int_{-k_1}^{-k_0} d\beta \times \\ &\times \int_{2(t-r)+\beta}^{t+r} (\alpha - \beta)^{1-(n-1)p/2} (t+r-\alpha)^{(n-3)/2} d\alpha \\ &\geq \frac{CC_g^p(t-r)^{(n-2)-(n-1)p/2}}{2^{(n-1)p-(3n-11)/2}r^{(3n-7)/2}} \varepsilon^p \int_{-k_1}^{-k_0} d\beta \times \\ &\times \int_{2(t-r)+\beta}^{3(t-r)} \{3(t-r)-\alpha\}^{(n-3)/2} d\alpha \end{aligned}$$

in  $\Sigma_0$ . The second term of the right-hand side of (7.5) follows from this inequality. Therefore, the proof of Lemma 7.3 is ended.  $\square$

**[The 2nd Step] Comparison argument.**

Let us consider a solution  $w$  of

$$\begin{aligned} w(t-r) = & \frac{C2^{(n-3)/2}(t-r)^{(n-1)/2}}{r^{(3n-7)/2}} \int_{2k}^{t-r} (t-r-\beta)^{(n-3)/2} d\beta \\ & \times \int_{2(t-r)+\beta}^{t+r} (t+r-\alpha)^{(n-3)/2} |w(\beta)|^p d\alpha \\ & + \frac{E_1(t-r)^{(3n-5)/2-(n-1)p/2}}{2r^{(3n-7)/2}} \varepsilon^p. \end{aligned} \quad (7.6)$$

Then we have the following comparison lemma.

**Lemma 7.4** *Let  $u$  be a solution of (7.2) and  $w$  be a solution of (7.6). Then,  $u$  and  $w$  satisfy*

$$u > w \quad \text{in } \Sigma_0.$$

**Proof.** Fix a point  $(r_0, t_0) \in \Sigma_0$ . Define

$$\Lambda(r, t) = \{(\lambda, \tau) \in D(r, t) : 2k \leq \tau - \lambda \leq \lambda\},$$

where we set

$$D(r, t) = \{(\lambda, \tau) : t-r \leq \tau + \lambda \leq t+r, -k \leq \tau - \lambda \leq t-r\}$$

which is the domain of the integral in (4.17). Let us consider  $u$  and  $v$  in  $\Lambda(r_0, t_0)$ . Note that  $u > w$  on  $\tau - \lambda = 2k$  and at  $(2k, 4k)$  which is an edge of  $\Sigma_0$ . By compactness of the closure of  $\Lambda(r_0, t_0)$ , we have  $u > w$  in a neighborhood of  $\tau - \lambda = 2k$  and  $\lambda \geq 2k$ .

Assume that there exist a point  $(r_1, t_1)$  with  $u(r_1, t_1) = w(t_1 - r_1)$ , which is nearest to  $(2k, 4k)$  in such a neighborhood. Since  $u > w$  in  $R(r_1, t_1)$ , we have

$$\begin{aligned} & \frac{C2^{(n-3)/2}(t_1 - r_1)^{(n-1)/2}}{r_1^{(3n-7)/2}} \iint_{R(r_1, t_1)} (t_1 - r_1 - \tau + \lambda)^{(n-3)/2} \times \\ & \times (t_1 + r_1 - \tau - \lambda)^{(n-3)/2} |u(\lambda, \tau)|^p d\lambda d\tau + \frac{E_1(t_1 - r_1)^{(3n-5)/2-(n-1)p/2}}{r_1^{(3n-7)/2}} \varepsilon^p \\ & > \frac{C2^{(n-3)/2}(t_1 - r_1)^{(n-1)/2}}{r_1^{(3n-7)/2}} \iint_{R(r_1, t_1)} (t_1 - r_1 - \tau + \lambda)^{(n-3)/2} \\ & \times (t_1 + r_1 - \tau - \lambda)^{(n-3)/2} |w(\tau - \lambda)|^p d\lambda d\tau + \frac{E_1(t_1 - r_1)^{(3n-5)/2-(n-1)p/2}}{2r_1^{(3n-7)/2}} \varepsilon^p, \end{aligned}$$

In view of (7.5) and (7.6), this inequality implies that  $u > w$  at  $(r_1, t_1)$ , which is a contradiction to the definition of  $(r_1, t_1)$ . Therefore, we have  $u > w$  in  $\Lambda(r_0, t_0)$ .  $(r_0, t_0)$  stands for any point in  $\Sigma_0$ , so that  $\Lambda(r_0, t_0)$  covers all of  $\Sigma_0$ . The proof is completed.  $\square$

We note that Lemma 7.4 implies that the lifespan of  $w$  is greater than the one of  $u$ , so that it is sufficient to look for the lifespan of  $w$  in  $\Sigma_0$ . By definition of  $w$  in (7.6), we have

$$w(\xi) \geq \frac{C\xi^{3-n}}{2^{n-2}} \int_{2k}^{\xi} (\xi - \beta)^{(n-3)/2} |w(\beta)|^p d\beta \\ \times \int_{2\xi+\beta}^{3\xi} (3\xi - \alpha)^{(n-3)/2} d\alpha + \frac{E_1}{2^{(3n-5)/2}} \xi^{-q-(n-1)/2} \varepsilon^p$$

in  $\Gamma_0$ , where we set

$$\xi = \frac{r}{2}, \quad \Gamma_0 = \{t - r = \xi, r \geq 4k\}.$$

Hence we obtain that

$$w(\xi) \geq \frac{C\xi^{3-n}}{2^{n-3}(n-1)} \int_{2k}^{\xi} (\xi - \beta)^{n-2} |w(\beta)|^p d\beta + \frac{E_1 \xi^{-q-(n-1)/2}}{2^{(3n-5)/2}} \varepsilon^p$$

for  $\xi \geq 2k$ . Then, it follows from the setting

$$W(\xi) = \xi^{q+(n-1)/2} w(\xi)$$

that

$$W(\xi) \geq D_n \xi^{q-(n-5)/2} \int_{2k}^{\xi} \frac{(\xi - \beta)^{n-2} |W(\beta)|^p d\beta}{\beta^{(n-1)p/2+pq}} + E_2 \varepsilon^p \quad \text{for } \xi \geq 2k, \quad (7.7)$$

where we set

$$D_n = \frac{C}{2^{n-3}(n-1)}, \quad E_2 = \frac{E_1}{2^{(3n-5)/2}}.$$

Therefore we obtain the iteration frame in this section,

$$W(\xi) \geq D_n \int_{2k}^{\xi} \left( \frac{\xi - \beta}{\xi} \right)^{n-2} \frac{|W(\beta)|^p}{\beta^{pq}} d\beta + E_2 \varepsilon^p \quad \text{for } \xi \geq 2k. \quad (7.8)$$

### [The 3rd step] Slicing method in the iteration.

Let us define a blow-up domain as follows. Let us set

$$\Gamma_j = \{\xi \geq l_j k\}, \quad l_j = 2 + \frac{1}{2} + \cdots + \frac{1}{2^j} \quad (j \in \mathbf{N}).$$

We shall use the fact that a sequence  $\{l_j\}$  is monotonously increasing and bounded,  $2 < l_j < 3$ , so that  $\Gamma_{j+1} \subset \Gamma_j$ . Assume an estimate of the form

$$W(\xi) \geq C_j \left( \log \frac{\xi}{l_j k} \right)^{a_j} \quad \text{in } \Gamma_j \quad (7.9)$$

where  $a_j \geq 0$  and  $C_j > 0$ . Putting (7.9) into (7.8) and recalling that  $pq = 1$ , we get an estimate in  $\Gamma_{j+1}$  such as

$$W(\xi) \geq D_n C_j^p \int_{l_j k}^{\xi} \left( \frac{\xi - \beta}{\xi} \right)^{n-2} \left( \log \frac{\beta}{l_j k} \right)^{pa_j} \frac{d\beta}{\beta}.$$

Noting that  $\frac{l_j}{l_{j+1}}\xi \geq l_j k$  in  $\Gamma_{j+1}$ , we have

$$\begin{aligned} W(\xi) &\geq D_n C_j^p \int_{l_j k}^{l_j \xi / l_{j+1}} \left( \frac{\xi - \beta}{\xi} \right)^{n-2} \left( \log \frac{\beta}{l_j k} \right)^{pa_j} \frac{d\beta}{\beta} \\ &\geq D_n C_j^p \left( 1 - \frac{l_j}{l_{j+1}} \right)^{n-2} \int_{l_j k}^{l_j \xi / l_{j+1}} \left( \log \frac{\beta}{l_j k} \right)^{pa_j} \frac{d\beta}{\beta} \\ &= \frac{D_n C_j^p}{pa_j + 1} \left( 1 - \frac{l_j}{l_{j+1}} \right)^{n-2} \left( \log \frac{\xi}{l_{j+1} k} \right)^{pa_j + 1}. \end{aligned}$$

By monotonicity of  $\{l_j\}$  and

$$1 - \frac{l_j}{l_{j+1}} = \frac{l_{j+1} - l_j}{l_{j+1}} = \frac{1}{2^{j+1} l_{j+1}} \geq \frac{1}{3 \cdot 2^{j+1}},$$

we finally obtain

$$W(\xi) \geq C_{j+1} \left( \log \frac{\xi}{l_{j+1} k} \right)^{pa_j + 1} \text{ in } \Gamma_{j+1}, \quad (7.10)$$

where we set

$$C_{j+1} = \frac{D_n C_j^p}{3^{n-2} \cdot 2^{(j+1)(n-2)} (pa_j + 1)}.$$

Now, we are in a position to define sequences in the iteration. In view of (7.8), the first estimate is  $W(\xi) \geq E_2 \varepsilon^p$ , so that, with the help of (7.9) and (7.10), a sequence  $\{a_j\}$  should be defined by

$$a_1 = 0, \quad a_{j+1} = pa_j + 1 \quad (j \in \mathbf{N}).$$

Also a sequence  $\{C_j\}$  should be defined by

$$C_1 = E_2 \varepsilon^p, \quad C_{j+1} = \frac{D_n C_j^p}{3^{n-2} \cdot 2^{(j+1)(n-2)} (pa_j + 1)} \quad (j \in \mathbf{N}).$$

One can easily check that

$$a_j = \frac{p^{j-1} - 1}{p - 1} \quad (j \in \mathbf{N}),$$

which gives us

$$\frac{1}{pa_j + 1} \geq \frac{p-1}{p^j}.$$

Thus one can find that

$$C_{j+1} \geq E \frac{C_j^p}{(2^{n-2}p)^j} \quad (j \in \mathbf{N}),$$

where  $E$  is a positive constant defined by

$$E = \frac{D_n(p-1)}{6^{n-2}}.$$

Hence we inductively obtain, for  $j \geq 2$ , that

$$\log C_j \geq p^{j-1} \left\{ \log C_1 + \sum_{k=0}^{j-2} \frac{p^k \log E - (j-1-k)p^k \log(2^{n-2}p)}{p^{j-1}} \right\}.$$

The sum part of above inequality converges as  $j \rightarrow \infty$  by d'Alembert's criterion. It follows from this fact that there exist a constant  $S$  independent of  $j$  such that

$$C_j \geq \exp\{p^{j-1}(\log C_1 + S)\} \quad \text{for } j \geq 2.$$

Combining all the estimates above and making use of the monotonicity of  $\Gamma_j$ , we have the final inequality

$$\begin{aligned} W(\xi) &\geq \exp\{p^{j-1}(\log C_1 + S)\} \left( \log \frac{\xi}{3k} \right)^{(p^{j-1}-1)/(p-1)} \\ &= \exp\{p^{j-1}I(\xi)\} \left( \log \frac{\xi}{3k} \right)^{-1/(p-1)} \end{aligned}$$

in  $\Gamma_\infty = \{\xi \geq 3k\}$ , where we set

$$I(\xi) = \log \left( e^S E_2 \varepsilon^p \left( \log \frac{\xi}{3k} \right)^{1/(p-1)} \right).$$

If there exist a point  $\xi_0 \in \Gamma_\infty \subset \Gamma_j$  ( $j \geq 1$ ) such that  $I(\xi_0) > 0$ , we get  $W(\xi_0) \rightarrow \infty$  as  $j \rightarrow \infty$ . Note that  $I(\xi_0) > 0$  is equivalent to

$$\xi_0 > 3k \exp\{(e^S E_2)^{-(p-1)} \varepsilon^{-p(p-1)}\}.$$

It is trivial that there exists a positive constant  $\varepsilon_0 = \varepsilon_0(g, n, p, k)$  such that

$$\exp\{(e^S E_2)^{-(p-1)} \varepsilon^{-p(p-1)}\} \geq 1 \quad \text{for } 0 < \varepsilon \leq \varepsilon_0.$$

Since there exists  $(r_0, t_0) \in \Sigma_0$  such that  $t_0 - r_0 = \xi_0 > 3k$ , we obtain the desired conclusion;

$$T > 3k \exp\{(e^S E_2)^{-(p-1)} \varepsilon^{-p(p-1)}\}.$$

Therefore the proof of the critical case is now completed with a minor modification on  $\varepsilon_0$ .  $\square$

## 8 Upper bound of the lifespan for the subcritical case in odd dimensions

Similarly to the previous section, we prove the blow-up result of solution for (2.17) in the subcritical case in odd space dimensions. Note that we do not have to make use of the slicing method.

**Proposition 8.1** *Suppose that the assumptions of Theorem 2.2 are fulfilled. Let  $u$  be a  $C^0$ -solution of (2.17) in  $\mathbf{R}^n \times [0, T]$ . Then, there exists a positive constant  $\varepsilon_0 = \varepsilon_0(g, n, p, k)$  such that  $T$  cannot be taken as*

$$T > c\varepsilon^{-2p(p-1)/\gamma(p,n)} \text{ if } 1 < p < p_0(n) \quad (8.1)$$

for  $0 < \varepsilon \leq \varepsilon_0$ , where  $c$  is a positive constant independent of  $\varepsilon$ .

**Proof.** Because of the fact that  $-(n-1)p/2 - pq < 0$  for  $n \geq 5$ , (7.7) yields

$$W(\xi) \geq D_n \xi^{-(n-2)-pq} \int_{2k}^{\xi} (\xi - \beta)^{n-2} |W(\beta)|^p d\beta + E_2 \varepsilon^p \quad \text{for } \xi \geq 2k. \quad (8.2)$$

This is our iteration frame in this case.

Assume an estimate of the form

$$W(\xi) \geq C_j \frac{(\xi - 2k)^{a_j}}{\xi^{b_j}} \text{ in } \Gamma_0, \quad (8.3)$$

where  $a_j, b_j \geq 0$  and  $C_j > 0$ . Then, putting (8.3) into (8.2), we get

$$W(\xi) \geq \frac{D_n C_j^p}{\xi^{(n-2)+pq+pb_j}} \int_{2k}^{\xi} (\xi - \beta)^{n-2} (\beta - 2k)^{pa_j} d\beta \quad \text{in } \Gamma_0.$$

Applying the integration by parts to  $\beta$ -integral  $(n-2)$  times, we obtain that

$$\begin{aligned} & \frac{n-2}{pa_j+1} \int_{2k}^{\xi} (\xi - \beta)^{n-3} (\beta - 2k)^{pa_j+1} d\beta \\ & \geq \frac{(n-2)(n-3)}{(pa_j+2)^2} \int_{2k}^{\xi} (\xi - \beta)^{n-4} (\beta - 2k)^{pa_j+2} d\beta \\ & \dots \\ & \geq \frac{(n-2)!}{(pa_j+n-1)^{n-1}} (\xi - 2k)^{pa_j+n-1}. \end{aligned}$$

Therefore we finally get

$$W(\xi) \geq C_{j+1} \frac{(\xi - 2k)^{pa_j+n-1}}{\xi^{pb_j+n-2+pq}} \text{ in } \Gamma_0, \quad (8.4)$$

where we set

$$C_{j+1} = \frac{D_n C_j^p (n-2)!}{(pa_j + n - 1)^{n-1}}.$$

Now, we are in a position to define sequences in the iteration. In view of (8.2), the first estimate is  $W(\xi) \geq E_2 \varepsilon^p$ , so that, with the help of (8.3) and (8.4), sequences  $\{a_j\}$  and  $\{b_j\}$  should be defined by

$$a_1 = 0, \quad a_{j+1} = pa_j + n - 1 \quad (j \in \mathbf{N})$$

and

$$b_1 = 0, \quad b_{j+1} = pb_j + n - 2 + pq \quad (j \in \mathbf{N}).$$

Also a sequence  $\{C_j\}$  should be defined by

$$C_1 = E_2 \varepsilon^p, \quad C_{j+1} = \frac{D_n C_j^p (n-2)!}{(pa_j + n - 1)^{n-1}} \quad (j \in \mathbf{N}).$$

One can readily check that

$$a_j = \frac{n-1}{p-1}(p^{j-1} - 1), \quad b_j = \frac{pq + n - 2}{p-1}(p^{j-1} - 1) \quad (j \geq \mathbf{N}),$$

which gives us

$$\frac{1}{pa_j + n - 1} \geq \frac{p-1}{p^j(n-1)}.$$

Hence one can find that

$$C_{j+1} \geq F \frac{C_j^p}{p^{(n-1)j}} \quad (j \in \mathbf{N}),$$

where  $F$  is a positive constant defined by

$$F = \frac{D_n(n-2)!(p-1)^{n-1}}{(n-1)^{n-1}}.$$

Due to the induction argument again, we obtain, for  $j \geq 2$ , that

$$\log C_j \geq p^{j-1} \left\{ \log C_1 + \sum_{k=0}^{j-2} \frac{p^k \log F - (j-1-k)p^k \log(p^{n-1})}{p^{j-1}} \right\}.$$

As before, this inequality yields that there exist a constant  $S$  independent of  $j$  such that

$$C_j \geq \exp\{p^{j-1}(\log C_1 + S)\} \quad \text{for } j \geq 2.$$

Combining all the estimates above, we reach the the final inequality

$$W(\xi) \geq \exp\{p^{j-1}I(\xi)\} \frac{\xi^{(pq+n-2)/(p-1)}}{(\xi - 2k)^{(n-1)/(p-1)}} \quad \text{for } \xi \geq 2k,$$

where we set

$$I(\xi) = \log \left( E_2 e^S \varepsilon^p (\xi - 2k)^{(n-1)/(p-1)} \xi^{-(pq+n-2)/(p-1)} \right).$$

Note that

$$\frac{n-1}{p-1} - \frac{pq+n-2}{p-1} = \frac{1-pq}{p-1}.$$

If there exist a point  $\xi_0 \in \{\xi \geq 4k\} \subset \{\xi \geq 2k\}$  such that  $I(\xi_0) > 0$ , the desired conclusion can be established by the same argument as in the previous section.  $I(\xi_0) > 0$  is equivalent to

$$\xi_0 > 2^{(n-1)/(1-pq)} (e^S E_2)^{-(p-1)/(1-pq)} \varepsilon^{-2p(p-1)/\gamma(p,n)}$$

in this case, so that the proof of the subcritical case is now completed.  $\square$

## 9 Upper bound of the lifespan for the critical case in even dimensions

In this section, we prove the blow-up theorem in the critical case in even dimensions. The proof is based on the one in odd dimensional case. However, Huygens' principle for  $u^0$  is no longer available. Therefore the blow-up domain to ensure the positivity of the linear part is modified.

**Proposition 9.1** *Suppose that of the assumption of Theorem 2.2 are fulfilled. Let  $u$  be a  $C^0$ -solution of (2.17) if  $n > 4$  and  $p = p_0(n)$ , or  $u$  be a classical solution of (2.18) if  $n = 4$  and  $p = p_0(4)$  in  $\mathbf{R}^n \times [0, T]$ . Then, there exists a positive constant  $\varepsilon_0 = \varepsilon_0(g, n, p, k)$  such that  $T$  cannot be taken as*

$$T > \exp \left( c \varepsilon^{-p(p-1)} \right) \quad \text{if } p = p_0(n) \tag{9.1}$$

for  $0 < \varepsilon \leq \varepsilon_0$ , where  $c$  is a positive constant independent of  $\varepsilon$ .



**Proof.** Similarly to the odd dimensional case, we may assume that the solution of (2.17) is radially symmetric without loss of the generality. Let  $u = u(r, t)$  be a  $C^0$ -solution of

$$u \geq \varepsilon u^0 + L_{even,1}(|u|^p) \quad \text{in} \quad (0, \infty) \times [0, T], \quad (9.2)$$

where  $L_{even,1}$  is defined by (4.18) and  $u^0 = u^0(r, t)$  is a solution of (7.3).

**[The 1st step] Estimate of  $u^0$ .**

We shall employ the following representation of  $u^0$ .

**Lemma 9.1 (Rammaha [27])** *Let  $n = 4, 6, 8, \dots$  and  $u^0$  be a solution of (7.3). Then,  $u^0$  is represented by*

$$u^0(r, t) = \frac{2}{\pi r^{(n-2)/2}} \int_0^t \frac{\rho d\rho}{\sqrt{t^2 - \rho^2}} \times \\ \times \int_{|r-\rho|}^{r+\rho} \frac{\lambda^{(n-2)/2} g(\lambda) T_{(n-4)/2}((\lambda^2 + r^2 - \rho^2)/(2r\lambda)) d\lambda}{\sqrt{\lambda^2 - (r - \rho)^2} \sqrt{(r + \rho)^2 - \lambda^2}},$$

where  $T_k$  is Tschebyscheff polynomials of degree  $k$  defined by

$$T_k(z) = \frac{(-1)^k}{(2k-1)!!} (1 - z^2)^{1/2} \frac{d^k}{dz^k} (1 - z^2)^{k-(1/2)}.$$

See (6b) on 681p. in [27] for the proof. This lemma implies the following estimate.

**Lemma 9.2 (Rammaha [27])** *Let  $n = 4, 6, 8, \dots$ . Assume (2.22). Then there exists a positive constant  $C_g$  such that, for  $t + k_0 < r < t + k_1$  and  $t \geq k_2$ ,*

$$u^0(r, t) \geq \frac{C_g}{r^{(n-1)/2}}, \quad (9.3)$$

where  $k_2 = k - k_0$ .

See Lemma 2 on 682p. in [27] for the proof.

Our frame in the iteration argument is obtained by the following lemma.

**Lemma 9.3** *Let  $u$  be a  $C^0$ -solution of (9.2). Assume (2.22). Then  $u$  in  $\Sigma_0 = \{(r, t) : 2k \leq t - r \leq r\}$  satisfies*

$$u(r, t) \geq \frac{C 2^{(n-1)/2} (t - r)^{(n-1)/2}}{(n-1) r^{(3n-5)/2}} \times \\ \times \iint_{R(r, t)} \{(t - r - \tau + \lambda)(t + r - \tau - \lambda)\}^{(n-2)/2} |u(\lambda, \tau)|^p d\lambda d\tau + \\ + \frac{F_1(t - r)^{(3n-3)/2 - (n-1)p/2}}{r^{(3n-5)/2}} \varepsilon^p + \varepsilon u^0(r, t), \quad (9.4)$$

where  $C$  is the one in (4.18) and

$$F_1 = \frac{CC_g^p(k_1 - k_0)2^{(11-3n)/2-(n-1)p}}{n(n-1)}.$$

**Proof.** In view of (4.18), we have that

$$\begin{aligned} L_{even,1}(|u|^p)(r, t) &\geq \frac{C}{r^{n-2}} \int_0^t (t-\tau)^{2-n} d\tau \int_{|t-r-\tau|}^{t+r-\tau} \lambda |u(\lambda, \tau)|^p d\lambda \times \\ &\quad \times \int_{|\lambda-r|}^{t-\tau} \frac{\rho h(\lambda, \rho, r)}{\sqrt{(t-\tau)^2 - \rho^2}} d\rho \end{aligned}$$

in  $\Sigma_0$ . Noticing that  $(\lambda + r)^2 - \rho^2 \geq (\lambda + r)^2 - (t - \tau)^2$  for  $\rho \leq t - \tau$ , we get

$$\begin{aligned} L_{even,1}(|u|^p)(r, t) &\geq \frac{C}{r^{n-2}} \int_0^t (t-\tau)^{2-n} d\tau \times \\ &\quad \times \int_{|t-r-\tau|}^{t+r-\tau} \frac{\lambda |u(\lambda, \tau)|^p \{(\lambda + r)^2 - (t - \tau)^2\}^{(n-3)/2} d\lambda}{\{(t - \tau)^2 - (\lambda - r)^2\}^{1/2}} \times \\ &\quad \times \int_{|\lambda-r|}^{t-\tau} \rho \{\rho^2 - (\lambda - r)^2\}^{(n-3)/2} d\rho \end{aligned}$$

in  $\Sigma_0$ . Since the  $\rho$ -integral above is

$$\frac{1}{n-1} \{(t - \tau)^2 - (\lambda - r)^2\}^{(n-1)/2},$$

we obtain that

$$\begin{aligned} L_{even,1}(|u|^p)(r, t) &\geq \frac{C}{r^{n-2}(n-1)} \int_0^t (t-\tau)^{2-n} d\tau \times \\ &\quad \times \int_{|t-r-\tau|}^{t+r-\tau} \{(\lambda + r)^2 - (t - \tau)^2\}^{(n-3)/2} \times \\ &\quad \times \{(t - \tau)^2 - (\lambda - r)^2\}^{(n-2)/2} \lambda |u(\lambda, \tau)|^p d\lambda \\ &\geq J_1 + J_2 \end{aligned}$$

in  $\Sigma_0$ , where we set

$$\begin{aligned} J_1(r, t) &= \frac{C}{(n-1)r^{n-2}} \int_{R(r,t)} (t-\tau)^{2-n} \{(\lambda + r)^2 - (t - \tau)^2\}^{(n-3)/2} \times \\ &\quad \times \{(t - \tau)^2 - (\lambda - r)^2\}^{(n-2)/2} \lambda |u(\lambda, \tau)|^p d\lambda d\tau, \\ J_2(r, t) &= \frac{C}{(n-1)r^{n-2}} \int_{S(r,t)} (t-\tau)^{2-n} \{(\lambda + r)^2 - (t - \tau)^2\}^{(n-3)/2} \times \\ &\quad \times \{(t - \tau)^2 - (\lambda - r)^2\}^{(n-2)/2} \lambda |u(\lambda, \tau)|^p d\lambda d\tau. \end{aligned}$$

Changing variables by (4.28) in  $J_1$ , we have that

$$J_1(r, t) \geq \frac{C}{2(n-1)r^{n-2}} \int_{2k}^{t-r} (t-r-\beta)^{(n-2)/2} (t+r-\beta)^{(n-3)/2} d\beta \times \\ \times \int_{2(t-r)+\beta}^{t+r} \{\alpha - (t-r)\}^{(n-3)/2} (t+r-\alpha)^{(n-2)/2} \times \\ \times \{t - (\alpha + \beta)/2\}^{2-n} (\alpha - \beta) |u(\lambda, \tau)|^p d\alpha$$

in  $\Sigma_0$ . Note that

$$t+r-\beta \geq 2r, \quad t - \frac{\alpha + \beta}{2} \leq r, \\ \alpha - \beta \geq 2(t-r), \quad \alpha - (t-r) \geq t-r+\beta \geq t-r$$

hold in the domain of the integral above. Hence we get

$$J_1(r, t) \geq \frac{C 2^{(n-3)/2} (t-r)^{(n-1)/2}}{(n-1)r^{(3n-5)/2}} \int_{2k}^{t-r} (t-r-\beta)^{(n-2)/2} d\beta \times \\ \times \int_{2(t-r)+\beta}^{t+r} (t+r-\alpha)^{(n-2)/2} |u(\lambda, \tau)|^p d\alpha$$

in  $\Sigma_0$ . Therefore, we obtain the first term of the right-hand side in (9.4).

Similarly to the above,  $J_2(r, t)$  is bounded from below by

$$\frac{C}{2(n-1)r^{n-2}} \int_{-k_1}^{-k_0} (t-r-\beta)^{(n-2)/2} (t+r-\beta)^{(n-3)/2} d\beta \\ \times \int_{2(t-r)+\beta}^{t+r} \{\alpha - (t-r)\}^{(n-3)/2} (t+r-\alpha)^{(n-2)/2} \\ \times \{t - (\alpha + \beta)/2\}^{2-n} (\alpha - \beta) |u(\lambda, \tau)|^p d\alpha$$

in  $\Sigma_0$ . Note that

$$t+r-\beta \geq r, \quad t - \frac{\alpha + \beta}{2} \leq 2r, \quad \alpha - (t-r) \geq t-r-k \\ \text{and } t-r-\beta \geq t-r$$

hold in the domain of the integral above. Hence (9.3) yields that  $J_2(r, t)$  in  $\Sigma_0$  is estimated from below by

$$\geq \frac{\varepsilon^p C C_g^p 2^{1-n} (t-r)^{(n-2)/2} (t-r-k)^{(n-3)/2}}{(n-1)r^{(3n-5)/2}} \int_{-k_1}^{-k_0} d\beta \\ \times \int_{2(t-r)+\beta}^{t+r} (\alpha - \beta)^{1-(n-1)p/2} (t+r-\alpha)^{(n-2)/2} d\alpha \\ \geq \frac{\varepsilon^p C C_g^p 2^{3-n-(n-1)p} (t-r)^{n/2-(n-1)p/2} (t-r-k)^{(n-3)/2}}{(n-1)r^{(3n-5)/2}} \int_{-k_1}^{-k_0} d\beta \times \\ \times \int_{2(t-r)+\beta}^{3(t-r)} \{3(t-r)-\alpha\}^{(n-2)/2} d\alpha.$$

The second term of the right-hand side of (9.4) follows from this inequality. Therefore, the proof of Lemma 9.3 is ended.  $\square$

Next, we shall show the positivity of the right-hand side of (9.4). Under the condition (2.22), (3.6) yields that

$$\begin{aligned}\varepsilon u^0(r, t) &\geq \frac{-C_{n,k,0,g}\varepsilon}{(t+r+2k)^{(n-1)/2}(t-r+2k)^{(n-1)/2}} \\ &\geq \frac{-C_{n,k,0,g}\varepsilon}{r^{(n-1)/2}(t-r)^{(n-1)/2}}\end{aligned}$$

for  $t-r \geq -k$ . Let us define a domain

$$\Sigma_1 = \left\{ (r, t) \in (0, \infty)^2 : r \geq t-r \geq \frac{r}{2}, r \geq K\varepsilon^{-L} \right\},$$

where we set

$$\begin{aligned}K &= \left( 2^{2n-(n-1)p/2-1} F_1^{-1} C_{n,k,0,g} \right)^{1/(n-(n-1)p/2)}, \\ L &= \frac{p-1}{n-(n-1)p/2} > 0.\end{aligned}$$

Taking  $\varepsilon$  to satisfy

$$K\varepsilon^{-L} \geq 4k$$

and setting

$$A(r, t) = \frac{F_1(t-r)^{(3n-3)/2-(n-1)p/2}}{r^{(3n-5)/2}} > 0,$$

we obtain that, in  $\Sigma_1$ ,

$$\begin{aligned}&\frac{A(r, t)}{2} \varepsilon^p + \varepsilon u^0(r, t) \\ &\geq \frac{F_1 \varepsilon^p (t-r)^{2n-2-(n-1)p/2} r^{(n-1)/2} - 2C_{n,k,0,g} \varepsilon r^{(3n-5)/2}}{2r^{2n-3}(t-r)^{(n-1)/2}} \geq 0.\end{aligned}$$

Making use of this inequality, we obtain that

$$\begin{aligned}u(r, t) &\geq \frac{C2^{(n-3)/2}(t-r)^{(n-1)/2}}{(n-1)r^{(3n-5)/2}} \int_{2k}^{t-r} (t-r-\beta)^{(n-2)/2} d\beta \times \\ &\quad \times \int_{2(t-r)+\beta}^{t+r} (t+r-\alpha)^{(n-2)/2} |u(\lambda, \tau)|^p d\alpha + \frac{A(r, t)}{2} \varepsilon^p\end{aligned}$$

in  $\Sigma_1$ . Cutting the domain of the integral, we get

$$\begin{aligned}u(r, t) &> \frac{C2^{(n-3)/2}(t-r)^{(n-1)/2}}{(n-1)r^{(3n-5)/2}} \int_{K\varepsilon^{-L}/2}^{t-r} (t-r-\beta)^{(n-2)/2} d\beta \times \\ &\quad \times \int_{3(t-r)}^{t+r} (t+r-\alpha)^{(n-2)/2} |u(\lambda, \tau)|^p d\alpha + \frac{A(r, t)}{4} \varepsilon^p\end{aligned}$$

in  $\Sigma_1$ . Here we introduce a change of variables  $(\alpha, \beta)$  to  $(\xi, \eta)$  by

$$\xi = \alpha, \quad \eta = \frac{\alpha + \beta}{2} - \frac{3}{2} \cdot \frac{\alpha - \beta}{2} = \frac{5\beta - \alpha}{4}.$$

Then, cutting the domain of the integral again, we get

$$\begin{aligned} u(r, t) &> \frac{C2^{(n+1)/2}(t-r)^{(n-1)/2}}{5(n-1)r^{(3n-5)/2}} \int_{K\varepsilon^{-L/2}}^{t-3r/2} \left\{ t - r - \left( \frac{4\eta + \xi}{5} \right) \right\}^{(n-2)/2} d\eta \times \\ &\times \int_{3(t-r)}^{t+r} (t+r-\xi)^{(n-2)/2} |u(\lambda, \tau)|^p d\xi + B(r, t)\varepsilon^p \end{aligned}$$

in  $\Sigma_2$ , where we set

$$\Sigma_2 = \left\{ (r, t) \in (0, \infty)^2 : \frac{r}{2} \geq t - \frac{3}{2}r \geq \frac{K\varepsilon^{-L}}{2} \right\}$$

and

$$B(r, t) = \frac{A(r, t)}{4}.$$

Therefore we obtain that, in  $\Sigma_2$ ,

$$\begin{aligned} u(r, t) &> \frac{C2^{(3n-3)/2}(t-r)^{(n-1)/2}}{5^{n/2}(n-1)r^{(3n-5)/2}} \int_{K\varepsilon^{-L/2}}^{t-3r/2} \left( t - \frac{3}{2}r - \eta \right)^{(n-2)/2} d\eta \times \\ &\times \int_{3(t-r)}^{t+r} (t+r-\alpha)^{(n-2)/2} |u(\lambda, \tau)|^p d\alpha + B(r, t)\varepsilon^p. \end{aligned} \tag{9.5}$$

### [The 2nd Step] Comparison argument.

Let us consider a solution  $y$  of

$$\begin{aligned} y \left( t - \frac{3}{2}r \right) &= \frac{C2^{(3n-3)/2}(t-r)^{(n-1)/2}}{5^{n/2}(n-1)r^{(3n-5)/2}} \int_{K\varepsilon^{-L/2}}^{t-3r/2} \left( t - \frac{3}{2}r - \eta \right)^{(n-2)/2} d\eta \times \\ &\times \int_{3(t-r)}^{t+r} (t+r-\alpha)^{(n-2)/2} |y(\eta)|^p d\alpha + B(r, t)\varepsilon^p. \end{aligned} \tag{9.6}$$

Then we have the following comparison lemma.

**Lemma 9.4** *Let  $u$  be a solution of (9.2) and  $y$  be a solution of (9.6). Then,  $u$  and  $y$  satisfy*

$$u > y \quad \text{in } \Sigma_2.$$

**Proof.** Fix a point for any  $(r_0, t_0) \in \Sigma_2$ . Define

$$\Lambda(r, t) = \left\{ (\lambda, \tau) \in D(r, t) : \frac{K\varepsilon^{-L}}{2} \leq \tau - \frac{3}{2}\lambda \leq \frac{\lambda}{2} \right\},$$

where

$$D(r, t) = \{(\lambda, \tau) : t - r \leq \tau + \lambda \leq t + r, -k \leq \tau - \lambda \leq t - r\}.$$

Let us consider  $u$  and  $y$  in  $\Lambda(r_0, t_0)$ . Note that  $u > y$  on  $\tau - \frac{3}{2}\lambda = \frac{K\varepsilon^{-L}}{2}$  and at  $(K\varepsilon^{-L}, 2K\varepsilon^{-L})$  which is an edge point of  $\Sigma_2$ . By compactness of the closure of  $\Lambda(r_0, t_0)$ , we have  $u > y$  in a neighborhood of  $\tau - \frac{3}{2}\lambda = \frac{K\varepsilon^{-L}}{2}$  and  $\lambda \geq K\varepsilon^{-L}$ .

Assume that there exist a point  $(r_1, t_1)$  with  $u(r_1, t_1) = y(t_1 - 3r_1/2)$  which is nearest to  $(K\varepsilon^{-L}, 2K\varepsilon^{-L})$  in such a neighborhood. Since  $u > y$  in  $R'(r_1, t_1)$ , we have

$$\begin{aligned} & \frac{C2^{(3n-5)/2}(t_1 - r_1)^{(n-1)/2}}{5^{(n-2)/2}(n-1)r_1^{(3n-5)/2}} \iint_{R'(r_1, t_1)} \left(t_1 - \frac{3}{2}r_1 - \tau + \frac{3}{2}\lambda\right)^{(n-2)/2} \times \\ & \quad \times (t_1 + r_1 - \tau - \lambda)^{(n-2)/2} |u(\lambda, \tau)|^p d\lambda d\tau + B(r_1, t_1)\varepsilon^p \\ & > \frac{C2^{(3n-5)/2}(t_1 - r_1)^{(n-1)/2}}{5^{(n-2)/2}(n-1)r_1^{(3n-5)/2}} \iint_{R'(r_1, t_1)} \left(t_1 - \frac{3}{2}r_1 - \tau + \frac{3}{2}\lambda\right)^{(n-2)/2} \times \\ & \quad \times \left(t_1 + r_1 - \tau - \lambda\right)^{(n-2)/2} \left|y\left(\tau - \frac{3}{2}\lambda\right)\right|^p d\lambda d\tau + B(r_1, t_1)\varepsilon^p, \end{aligned}$$

where we set

$$R'(r, t) = \left\{ (\lambda, \tau) : 3(t - r) \leq \tau + \lambda \leq t + r, \frac{K\varepsilon^{-L}}{2} \leq \tau - \frac{3}{2}\lambda \leq t - \frac{3}{2}r \right\}.$$

In view of (9.5) and (9.6), this inequality yield that  $u > y$  at  $(r_1, t_1)$ , which is a contradiction to the definition of  $(r_1, t_1)$ . Therefore, the proof of Lemma 9.4 is now established by the same argument as the one for Lemma 7.4.  $\square$

We note that Lemma 9.4 implies that the lifespan of  $y$  is greater than the lifespan of  $u$ , so that it is sufficient to look for the lifespan of  $y$  in  $\Sigma_2$ . By definition  $y$  in (9.6), we have

$$\begin{aligned} y(\xi) &= \frac{C3^{(n-1)/2}\xi^{2-n}}{2^{(3n-7)/2}5^{n/2}(n-1)} \int_{K\varepsilon^{-L}/2}^{\xi} (\xi - \eta)^{(n-2)/2} |y(\eta)|^p d\eta \\ & \quad \times \int_{9\xi}^{11\xi} (11\xi - \alpha)^{(n-2)/2} d\alpha + \frac{\varepsilon^p F_1 \xi^{-q-(n-1)/2}}{2^{(3n-3)}3^{(n-1)p/2-(3n-3)/2}} \end{aligned}$$

in  $\Gamma_1$ , where we set

$$\xi = \frac{r}{4}, \quad \Gamma_1 = \left\{ t - \frac{3}{2}r = \xi, r \geq 2K\varepsilon^{-L} \right\} \subset \Sigma_2.$$

Hence, we obtain that

$$y(\xi) \geq \frac{C3^{(n-1)/2}\xi^{(4-n)/2}}{2^{(2n-9)/2}5^{n/2}n(n-1)} \int_{K\varepsilon^{-L}/2}^{\xi} (\xi - \eta)^{(n-2)/2} |y(\eta)|^p d\eta \\ + \frac{\varepsilon^p F_1 \xi^{-q-(n-1)/2}}{2^{(3n-3)}3^{(n-1)p/2-(3n-3)/2}}$$

for  $\xi \geq K\varepsilon^{-L}/2$ . Then it follows from the setting

$$Y(\xi) = \xi^{q+(n-1)/2} y(\xi)$$

that

$$Y(\xi) \geq E_n \xi^{q+3/2} \int_{K\varepsilon^{-L}/2}^{\xi} \frac{(\xi - \eta)^{(n-2)/2} |Y(\eta)|^p d\eta}{\eta^{(n-1)p/2+pq}} + F_2 \varepsilon^p, \quad (9.7)$$

where we set

$$E_n = \frac{C3^{(n-1)/2}}{2^{(2n-9)/2}5^{n/2}n(n-1)}, \quad F_2 = \frac{F_1}{2^{(3n-3)}3^{(n-1)p/2-(3n-3)/2}}.$$

Therefore we obtain the iteration frame in this section,

$$Y(\xi) \geq E_n \int_{K\varepsilon^{-L}/2}^{\xi} \left( \frac{\xi - \eta}{\xi} \right)^{(n-2)/2} \frac{|Y(\eta)|^p}{\eta^{pq}} d\eta + F_2 \varepsilon^p \quad \text{for } \xi \geq \frac{K\varepsilon^{-L}}{2}. \quad (9.8)$$

### [The 3rd step] Slicing method with the iteration.

Let us define a blow-up domain as follows. Let us set

$$\Gamma_j = \{\xi \geq l_j K\varepsilon^{-L}\}, \quad l_j = \frac{1}{2} + \cdots + \frac{1}{2^j} \quad (j \in \mathbf{N}).$$

We shall use the fact that a sequence  $\{l_j\}$  is monotonously increasing and bounded as  $\frac{1}{2} < l_j < 1$ , so that  $\Gamma_{j+1} \subset \Gamma_j$ . Assume an estimate of the form

$$Y(\xi) \geq C_j \left( \log \frac{\xi}{l_j K\varepsilon^{-L}} \right)^{a_j} \quad \text{in } \Gamma_j, \quad (9.9)$$

where  $a_j \geq 0$  and  $C_j > 0$ . Putting (9.9) into (9.8) and recalling  $pq = 1$ , we get an estimate in  $\Gamma_{j+1}$  such as

$$Y(\xi) \geq E_n C_j^p \int_{l_j K\varepsilon^{-L}}^{\xi} \left( \frac{\xi - \eta}{\xi} \right)^{(n-2)/2} \left( \log \frac{\eta}{l_j K\varepsilon^{-L}} \right)^{pa_j} \frac{d\eta}{\eta}.$$

Noting that  $\frac{l_j}{l_{j+1}}\xi \geq l_j K \varepsilon^{-L}$  in  $\Gamma_{j+1}$ , we have

$$\begin{aligned} Y(\xi) &\geq E_n C_j^p \left(1 - \frac{l_j}{l_{j+1}}\right)^{(n-2)/2} \int_{l_j K \varepsilon^{-L}}^{l_j \xi / l_{j+1}} \left(\log \frac{\eta}{l_j k}\right)^{pa_j} \frac{d\eta}{\eta} \\ &= \frac{E_n C_j^p}{pa_j + 1} \left(1 - \frac{l_j}{l_{j+1}}\right)^{(n-2)/2} \left(\log \frac{\xi}{l_{j+1} K \varepsilon^{-L}}\right)^{pa_j+1}. \end{aligned}$$

By monotonicity of  $\{l_j\}$  and

$$1 - \frac{l_j}{l_{j+1}} = \frac{l_{j+1} - l_j}{l_{j+1}} = \frac{1}{2^{j+1} l_{j+1}} \geq \frac{1}{2^{j+1}},$$

we finally obtain

$$Y(\xi) \geq C_{j+1} \left(\log \frac{\xi}{l_{j+1} K \varepsilon^{-L}}\right)^{pa_j+1} \text{ in } \Gamma_{j+1}, \quad (9.10)$$

where we set

$$C_{j+1} = \frac{E_n C_j^p}{2^{(n-2)(j+1)/2} (pa_j + 1)}.$$

Now, we are in a position to define sequences in the iteration. In view of (9.8), the first estimate is  $Y(\xi) \geq F_2 \varepsilon^p$ , so that, with the help of (9.9) and (9.10), a sequence  $\{a_j\}$  should be defined by

$$a_1 = 0, \quad a_{j+1} = pa_j + 1 \quad (j \in \mathbf{N}).$$

Also a sequence  $\{C_j\}$  should be defined by

$$C_1 = F_2 \varepsilon^p, \quad C_{j+1} = \frac{E_n C_j^p}{2^{(n-2)(j+1)/2} (pa_j + 1)} \quad (j \in \mathbf{N}).$$

One can easily check that

$$a_j = \frac{p^{j-1} - 1}{p - 1} \quad (j \in \mathbf{N})$$

which gives us

$$\frac{1}{pa_j + 1} \geq \frac{p - 1}{p^j}.$$

Thus one can find that

$$C_{j+1} \geq E \frac{C_j^p}{(2^{(n-2)/2} p)^j} \quad (j \in \mathbf{N}),$$



where  $E$  is a positive constant defined by

$$E = \frac{E_n(p-1)}{2^{(n-2)/2}}.$$

Hence, we inductively obtain, for  $j \geq 2$ , that

$$\log C_j \geq p^{j-1} \left\{ \log C_1 + \sum_{k=0}^{j-2} \frac{p^k \log E - (j-1-k)p^k \log(2^{(n-2)/2}p)}{p^{j-1}} \right\}.$$

This inequality yields that there exist a constant  $S$  independent of  $j$  such that

$$C_j \geq \exp\{p^{j-1}(\log C_1 + S)\} \quad \text{for } j \geq 2.$$

Combining all the estimates above and making use of the monotonicity of  $\Gamma_j$ , we obtain the final inequality,

$$\begin{aligned} Y(\xi) &\geq \exp\{p^{j-1}(\log C_1 + S)\} \left( \log \frac{\xi}{K\varepsilon^{-L}} \right)^{(p^{j-1}-1)/(p-1)} \\ &= \exp\{p^{j-1}I(\xi)\} \left( \log \frac{\xi}{K\varepsilon^{-L}} \right)^{-1/(p-1)}. \end{aligned}$$

in  $\Gamma_\infty = \{\xi \geq K\varepsilon^{-L}\}$ , where we set

$$I(\xi) = \log \left( e^S F_2 \varepsilon^p \left( \log \frac{\xi}{K\varepsilon^{-L}} \right)^{1/(p-1)} \right).$$

If there exist a point  $\xi_0 \in \Gamma_\infty \subset \Gamma_j$  ( $j \geq 1$ ) such that  $I(\xi_0) > 0$ , we get the desired conclusion by the same argument in the end of section 7. In this case,  $I(\xi_0) > 0$  is equivalent to

$$\xi_0 > \exp\{(e^S F_2)^{-(p-1)} \varepsilon^{-p(p-1)}\} K\varepsilon^{-L}.$$

Therefore the proof of the critical case is now completed.  $\square$

## 10 Upper bound of the lifespan for the subcritical case in even dimensions

Similarly to the previous section, we prove the blow-up result of solution for (2.17) in the subcritical case in even space dimensions. Note that we do not have to make use of the slicing method.

**Proposition 10.1** *Suppose that the same assumption of Theorem 2.2 are fulfilled. Let  $u$  be a  $C^0$ -solution of (2.17)  $\mathbf{R}^n \times [0, T]$ . Then there exists a positive constant  $\varepsilon_0 = \varepsilon_0(g, n, p, k)$  such that  $T$  cannot be taken as*

$$T > c\varepsilon^{-2p(p-1)/\gamma(p,n)} \text{ if } 1 < p < p_0(n) \quad (10.1)$$

for  $0 < \varepsilon \leq \varepsilon_0$ , where  $c$  is a positive constant independent of  $\varepsilon$ .

**Proof.** Because of the fact that  $-(n-1)p/2 - pq < 0$  for  $n \geq 4$ , (9.7) yields

$$Y(\xi) \geq E_n \xi^{-(n-2)/2-pq} \int_{K\varepsilon^{-L}/2}^{\xi} (\xi - \eta)^{(n-2)/2} |Y(\eta)|^p d\eta + F_2 \varepsilon^p \quad (10.2)$$

for  $\xi \geq K\varepsilon^{-L}/2$ . This is our iteration frame in this case.

Assume an estimate of the form

$$Y(\xi) \geq C_j \frac{\left(\xi - \frac{K\varepsilon^{-L}}{2}\right)^{a_j}}{\xi^{b_j}} \text{ in } \Gamma_1, \quad (10.3)$$

where  $a_j, b_j \geq 0$  and  $C_j > 0$ . Then, putting (10.3) into (10.2), we get the following estimate in  $\Gamma_1$  of the form

$$Y(\xi) \geq E_n C_j^p \xi^{-(n-2)/2-pq-pb_j} \int_{K\varepsilon^{-L}/2}^{\xi} (\xi - \eta)^{(n-2)/2} \left(\eta - \frac{K\varepsilon^{-L}}{2}\right)^{pa_j} d\eta.$$

Applying the integration by parts to  $\beta$ -integral  $(n-2)/2$  times, we obtain that

$$\begin{aligned} & \frac{(n-2)}{2(pa_j+1)} \int_{K\varepsilon^{-L}/2}^{\xi} (\xi - \eta)^{(n-4)/2} \left(\eta - \frac{K\varepsilon^{-L}}{2}\right)^{pa_j+1} d\eta \\ & \geq \frac{(n-2)(n-4)}{(pa_j+2)^2 \cdot 2} \int_{K\varepsilon^{-L}/2}^{\xi} (\xi - \eta)^{(n-6)/2} \left(\eta - \frac{K\varepsilon^{-L}}{2}\right)^{pa_j+2} d\eta \\ & \dots \\ & \geq \frac{((n-2)/2)!}{\left(pa_j + \frac{n}{2}\right)^{n/2}} \left(\xi - \frac{K\varepsilon^{-L}}{2}\right)^{pa_j+n/2}. \end{aligned}$$

Therefore we finally get

$$Y(\xi) \geq \frac{C_{j+1} \left(\xi - \frac{K\varepsilon^{-L}}{2}\right)^{pa_j+n/2}}{\xi^{pb_j+(n-2)/2+pq}} \text{ in } \Gamma_1, \quad (10.4)$$

where

$$C_{j+1} = \frac{((n-2)/2)!E_n C_j^p}{\left(pa_j + \frac{n}{2}\right)^{n/2}}.$$

Now, we are in a position to define sequences in the iteration. In view of (10.2), the first estimate is  $Y(\xi) \geq F_2 \varepsilon^p$ , so that, with the help of (10.3) and (10.4), sequences  $\{a_j\}$  and  $\{b_j\}$  should be defined by

$$a_1 = 0, \quad a_{j+1} = pa_j + \frac{n}{2} \quad (j \in \mathbf{N})$$

and

$$b_1 = 0, \quad b_{j+1} = pb_j + pq + \frac{n-2}{2} \quad (j \in \mathbf{N}).$$

Also a sequence  $\{C_j\}$  should be defined by

$$C_1 = F_2 \varepsilon^p, \quad C_{j+1} = \frac{((n-2)/2)!E_n C_j^p}{\left(pa_j + \frac{n}{2}\right)^{n/2}} \quad (j \in \mathbf{N}).$$

One can readily check that

$$a_j = \frac{n}{2(p-1)}(p^{j-1} - 1), \quad b_j = \frac{pq + (n-2)/2}{p-1}(p^{j-1} - 1) \quad (j \in \mathbf{N}),$$

which gives us

$$\frac{1}{pa_j + n/2} \geq \frac{2(p-1)}{p^j n}.$$

Hence one can find that

$$C_{j+1} \geq F \frac{C_j^p}{p^{(n/2)j}} \quad (j \in \mathbf{N}),$$

where  $F$  is a positive constant defined by

$$F = \frac{E_n((n-2)/2)! \{2(p-1)\}^{n/2}}{n^{n/2}}.$$

Due to the induction argument again, we obtain, for  $j \geq 2$ , that

$$\log C_j \geq p^{j-1} \left\{ \log C_1 + \sum_{k=0}^{j-2} \frac{p^k \log F - (j-1-k)p^k \log(p^{n/2})}{p^{j-1}} \right\}.$$

This inequality yields that there exist a constant  $S$  independent of  $j$  such that

$$C_j \geq \exp\{p^{j-1}(\log C_1 + S)\} \quad \text{for } j \geq 2.$$

Combining all the estimates, we reach the final inequality

$$Y(\xi) \geq \exp\{p^{j-1}I(\xi)\} \frac{\xi^{(pq+(n-2)/2)/(p-1)}}{\left(\xi - \frac{K\varepsilon^{-L}}{2}\right)^{n/2(p-1)}} \quad \text{for } \xi \geq \frac{K\varepsilon^{-L}}{2},$$

where we set

$$I(\xi) = \log \left( F_2 e^S \varepsilon^p \left( \xi - \frac{K\varepsilon^{-L}}{2} \right)^{n/2(p-1)} \xi^{-(pq+(n-2)/2)/(p-1)} \right).$$

Note that

$$\frac{n}{2(p-1)} - \frac{pq + (n-2)/2}{p-1} = \frac{1-pq}{p-1}.$$

If there exist a point  $\xi_0 \in \{\xi \geq K\varepsilon^{-L}\} \subset \{\xi \geq (K\varepsilon^{-L})/2\}$  such that  $I(\xi_0) > 0$ , we get the desired conclusion as before.  $I(\xi_0) > 0$  is equivalent to

$$\xi_0 > 2^{n/2(1-pq)} (e^S F_2)^{-(1-p)/(1-pq)} \varepsilon^{-2p(p-1)/\gamma(p,n)}.$$

Therefore the proof of the subcritical case is completed.  $\square$

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